FREQUENCY-FILTERED PHOTON CORRELATIONS OF A THREE-LEVEL LADDER-TYPE ATOM

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MULTIPHOTONICS 2025

8TH OCTOBER, 2025

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Phys. Rev. Lett. **119**, 140504 (2017).

Phys. Rev. A **100,** 033802 (2019).

THREE-LEVEL LADDER-TYPE ATOM

- Hamiltonian

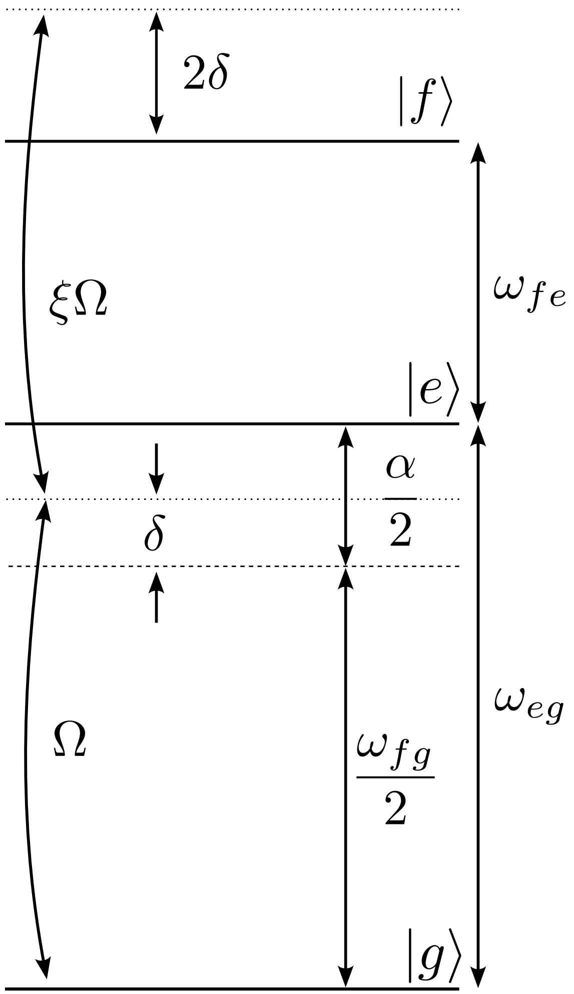
$$H_A = -\hbar\left(\frac{\alpha}{2}+\delta\right)|e\rangle\langle e| - 2\hbar\delta|f\rangle\langle f| + \hbar\frac{\Omega}{2}\Big(|e\rangle\langle g| + |g\rangle\langle e|\Big)$$
$$+\,\xi\frac{\Omega}{2}\Big(|f\rangle\langle e| + |e\rangle\langle f|\Big)$$

- Master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{i\hbar}[H_A,\rho] + \frac{\Gamma}{2}\Lambda(\Sigma_-)\rho, \quad \Sigma_- = |g\rangle\langle e| + \xi\,|e\rangle\langle f|$$

- α – anharmonicity
- δ - drive detuning from two-photon resonance
- ξ - dipole moment ratio
- Γ - atomic decay rate

$$\Lambda(X)\bullet = 2X\bullet X^\dagger - X^\dagger X\bullet - \bullet X^\dagger X$$



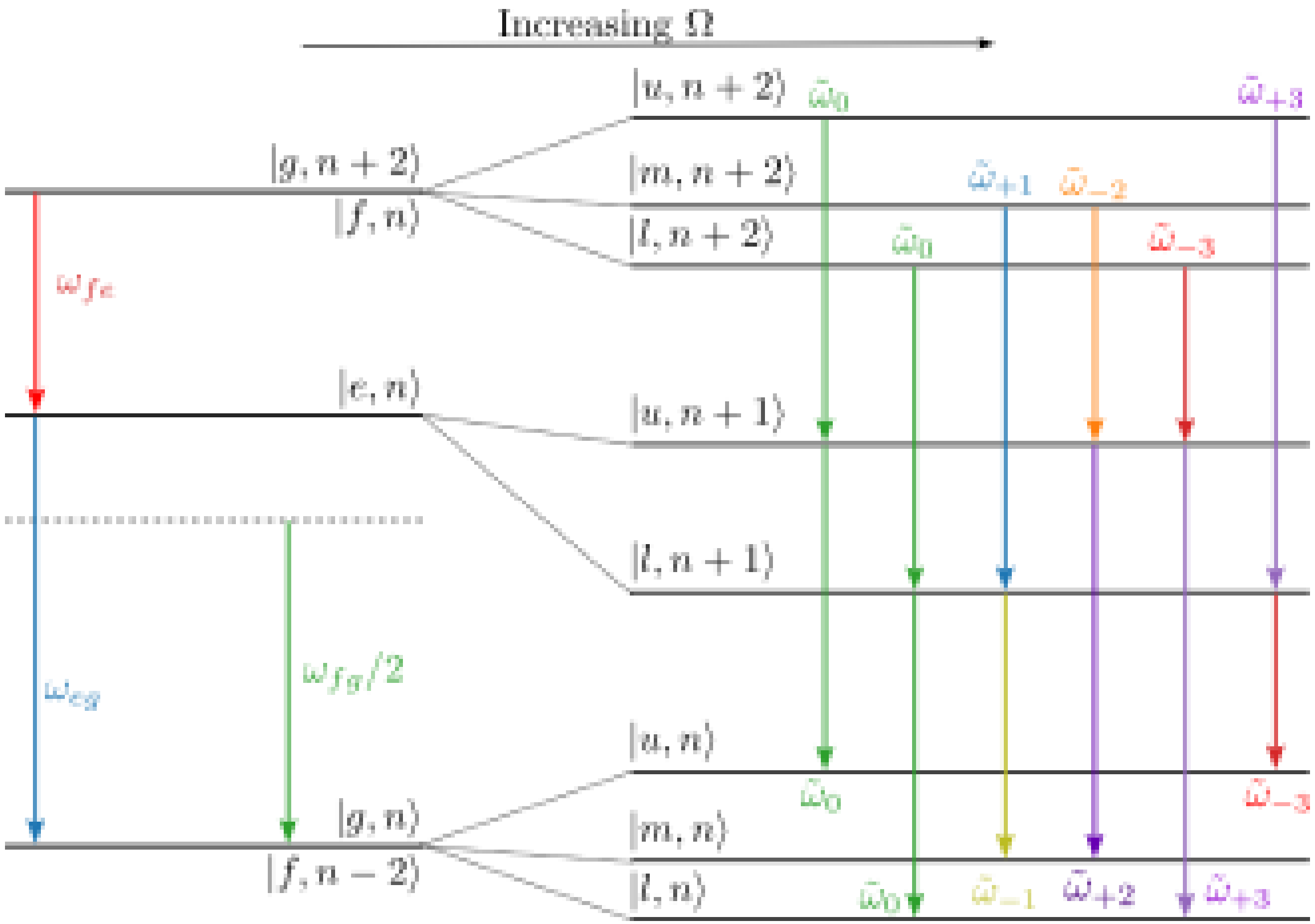
DRESSED STATES

Three dressed states ($\delta = 0$)

$$H_A|u\rangle = \hbar\omega_u|u\rangle$$
$$H_A|m\rangle = \hbar\omega_m|m\rangle$$
$$H_A|l\rangle = \hbar\omega_l|l\rangle$$

Seven transition frequencies:

$$\begin{aligned}\tilde{\omega}_0 &= \omega_d, \\ \tilde{\omega}_{\pm 1} &= \omega_d \pm (\omega_m - \omega_l)\,, \\ \tilde{\omega}_{\pm 2} &= \omega_d \pm (\omega_u - \omega_m)\,, \\ \tilde{\omega}_{\pm 3} &= \omega_d \pm (\omega_u - \omega_l)\,.\end{aligned}$$



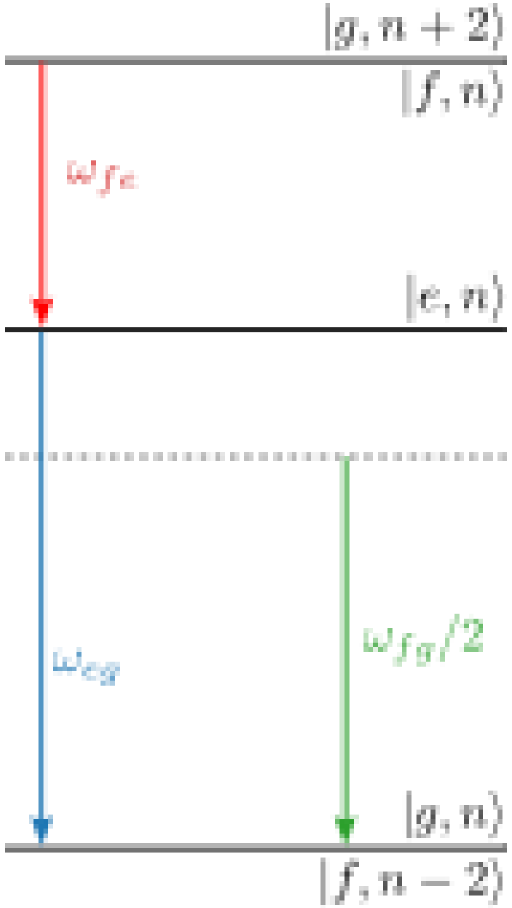
FLUORESCENCE SPECTRUM

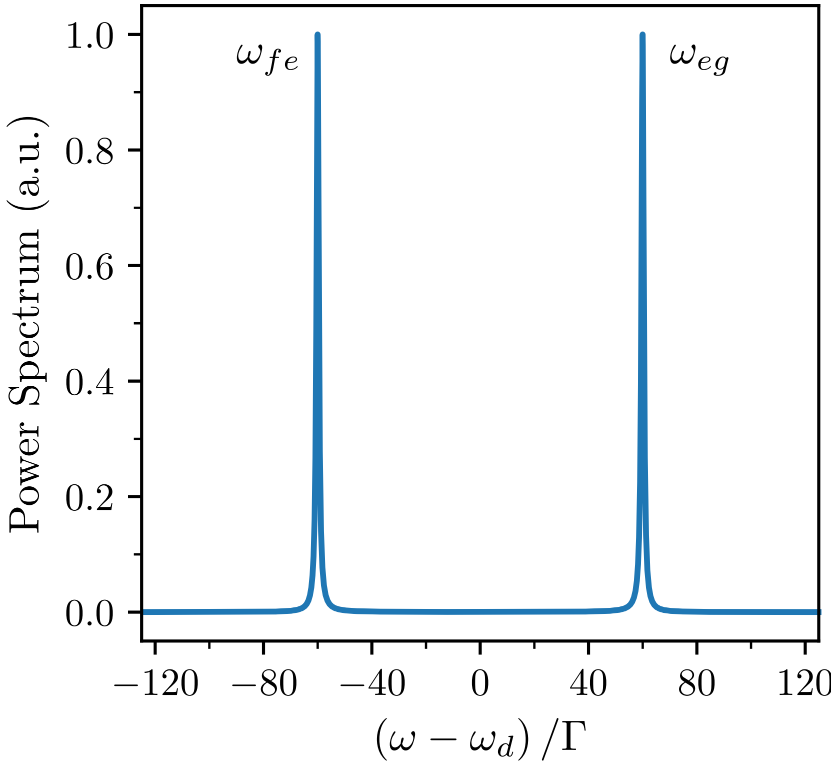
Incoherent power spectrum

$$S_{\mathrm{inc}}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau \langle \Delta\Sigma_{+}(\tau)\Delta\Sigma_{-}(0)\rangle$$

Fluctuation operators

$$\Delta\Sigma_{\pm} = \Sigma_{\pm} - \langle \Sigma_{\pm}\rangle_{ss}$$





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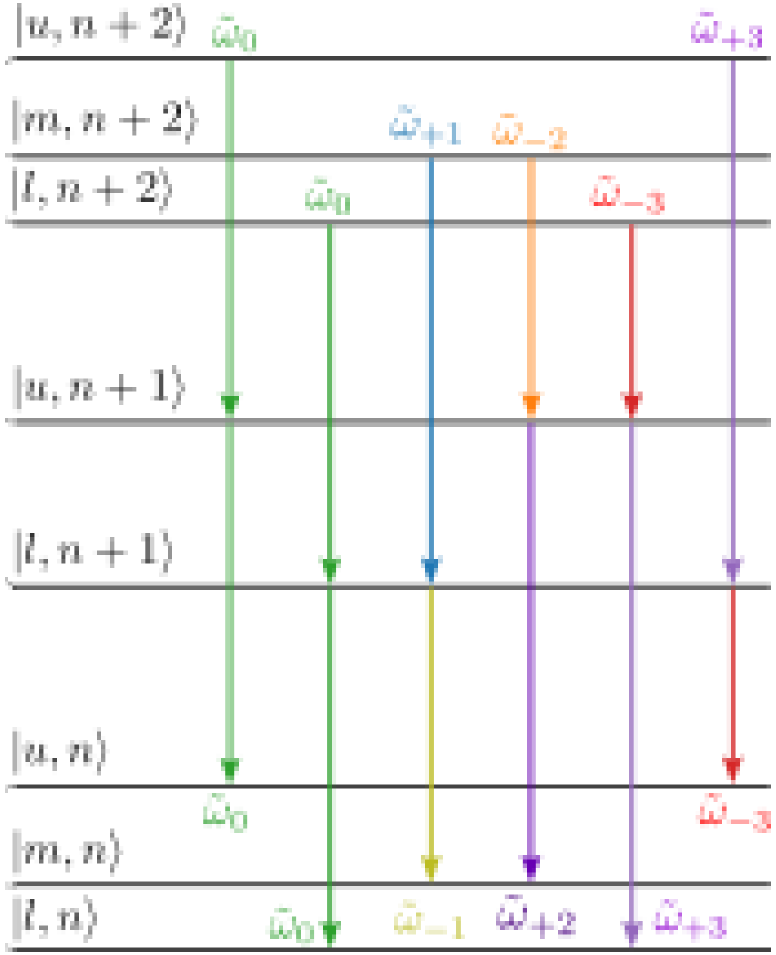
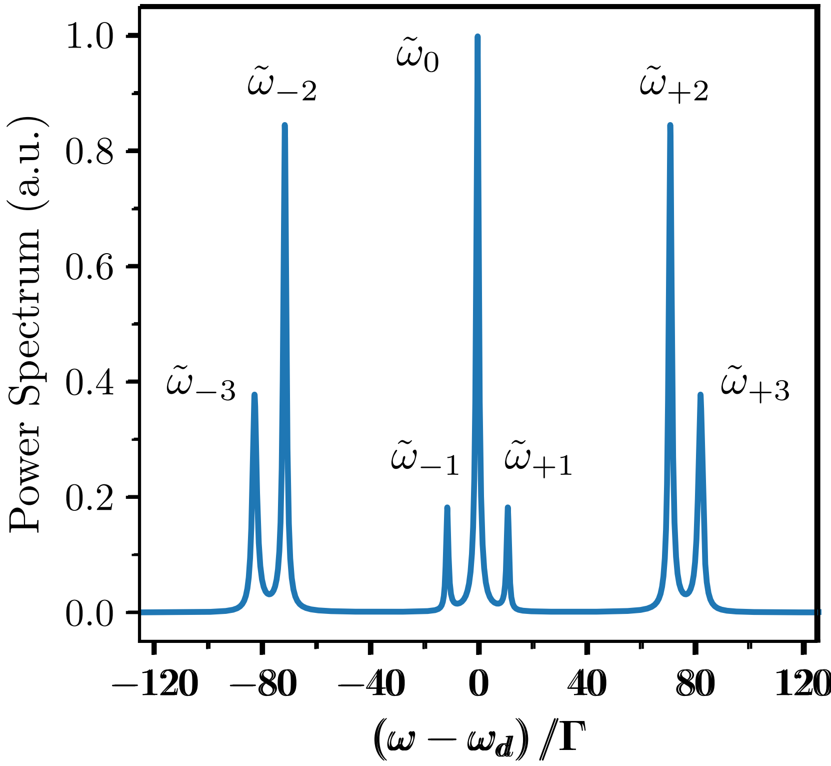
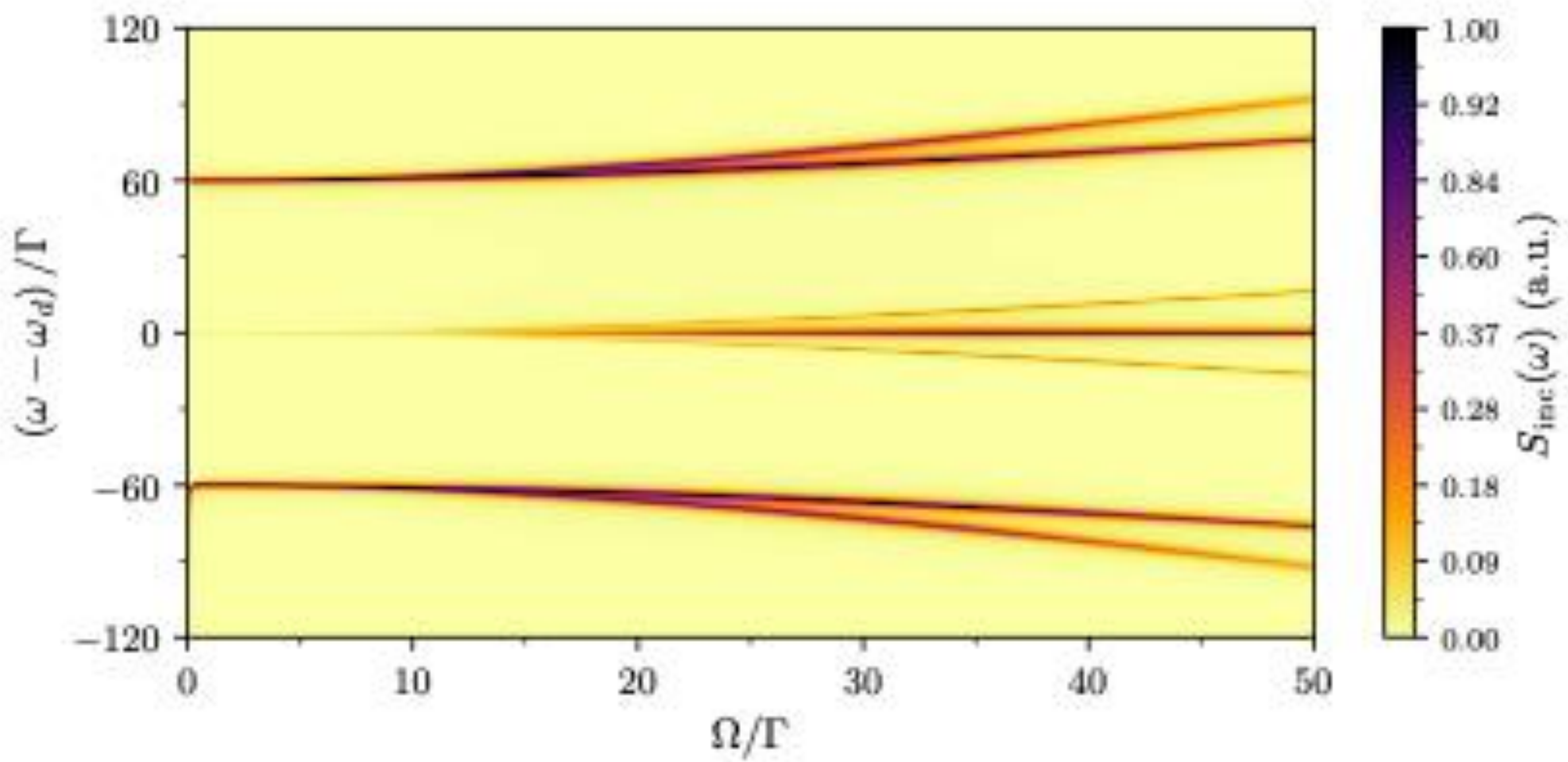




FLUORESCENCE SPECTRUM



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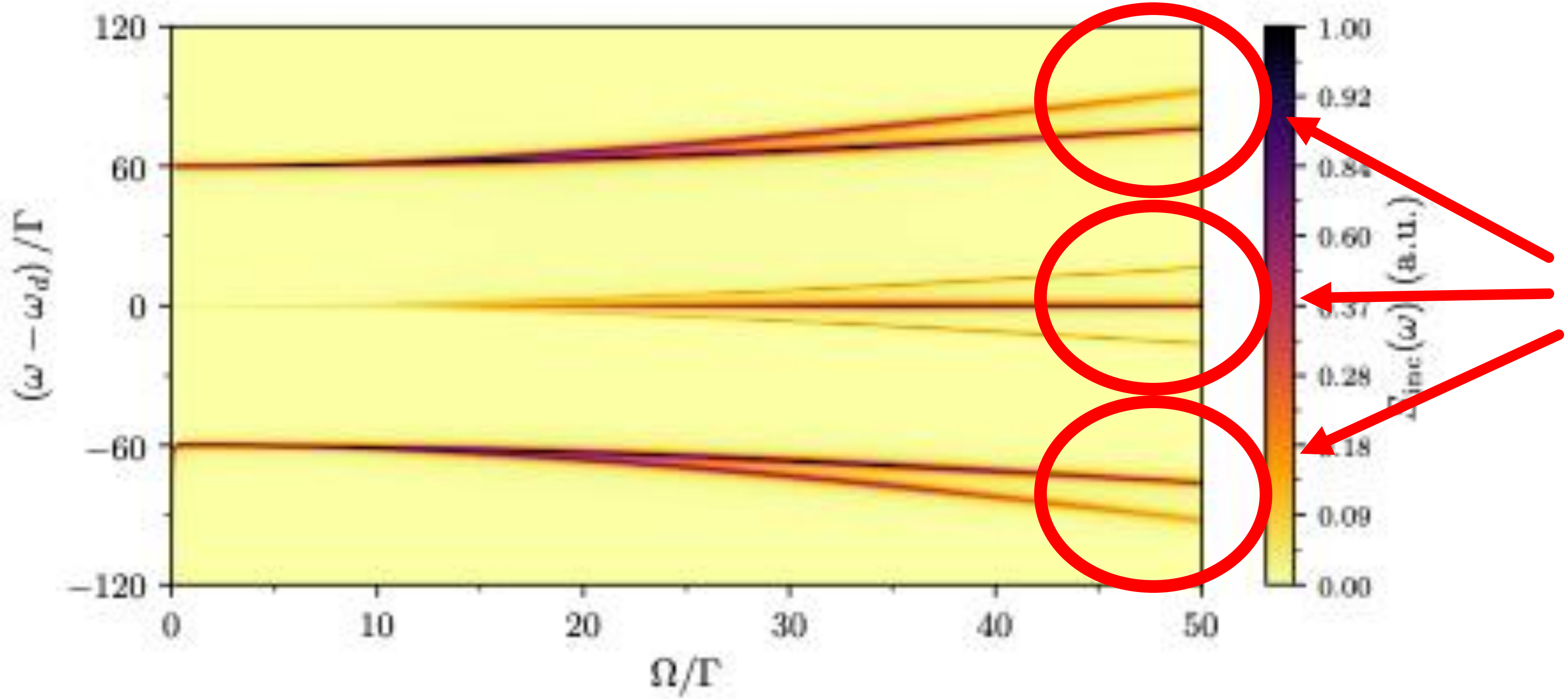

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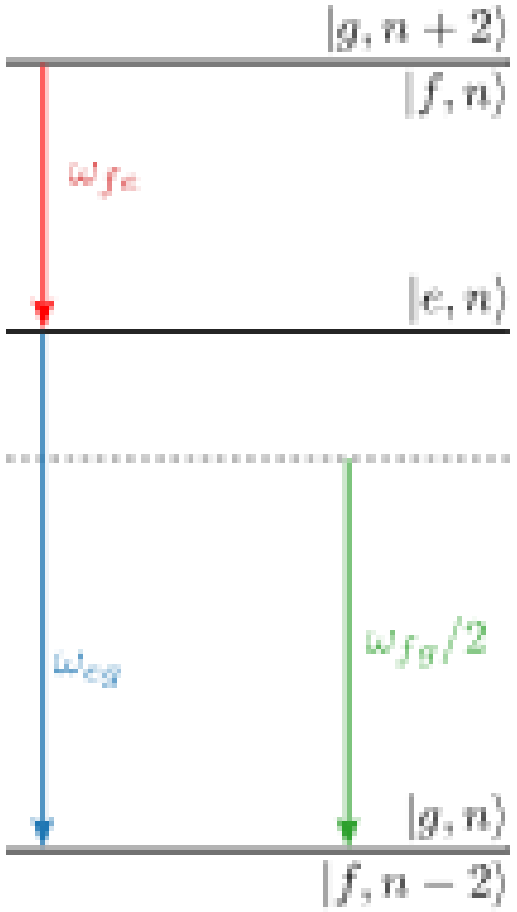
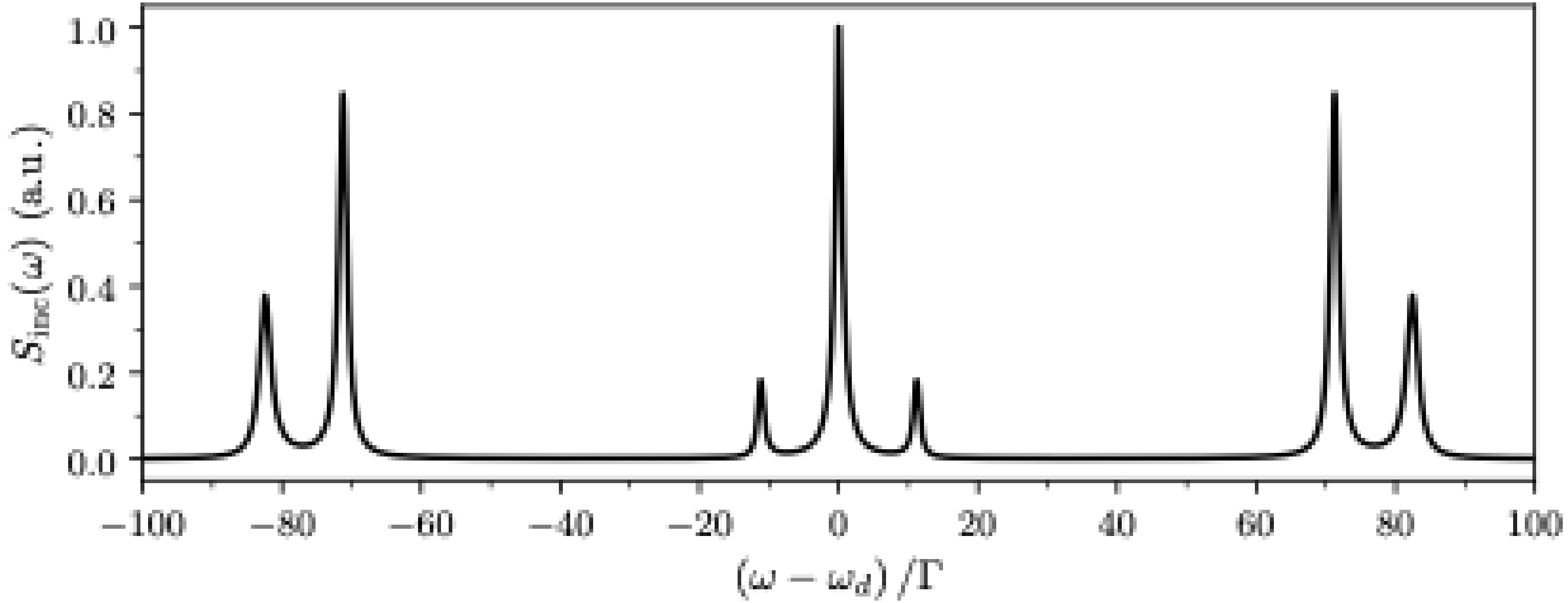
FLUORESCENCE SPECTRUM



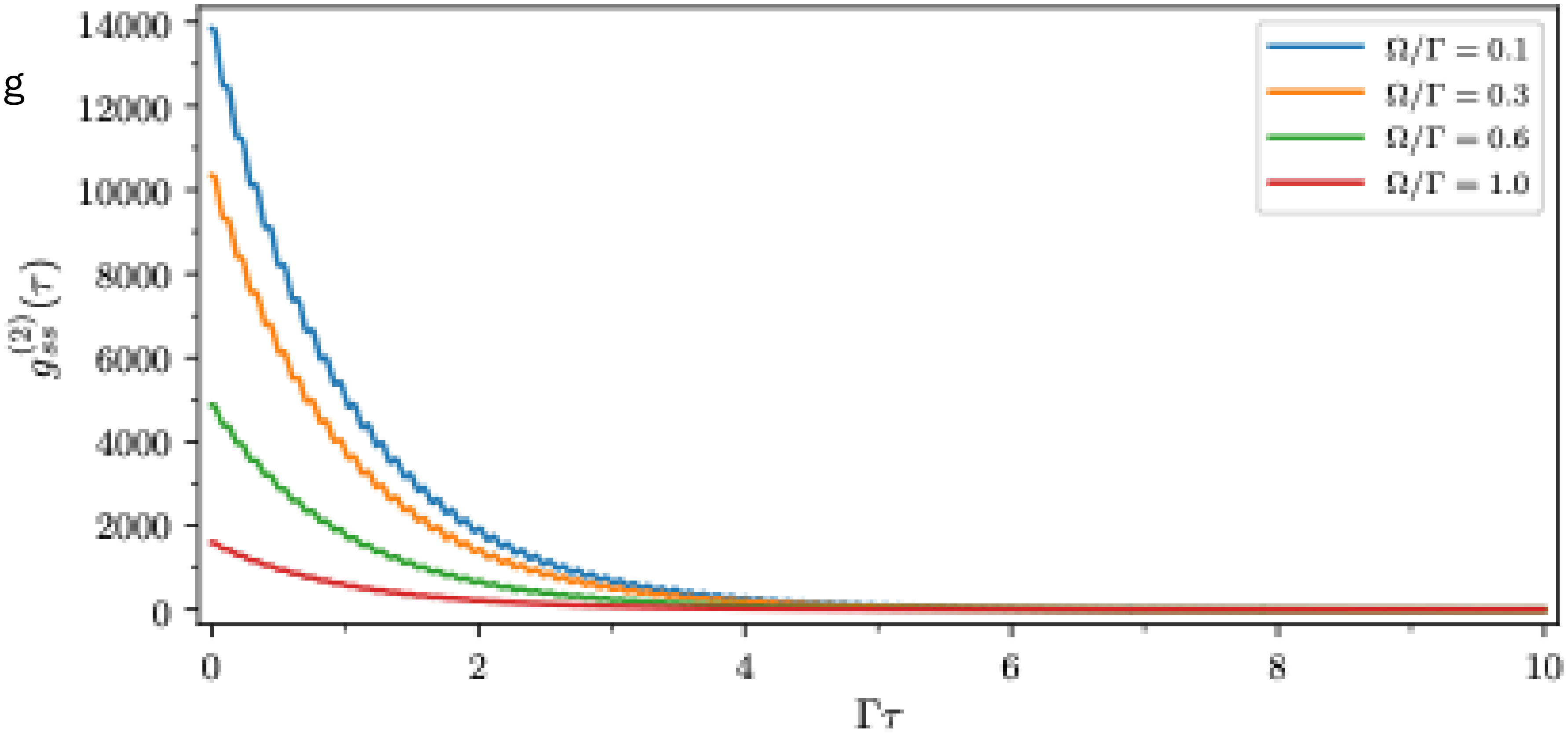
“Mollow” triplet from effective two-level atom

Stark shift induced doublets



PHOTON CORRELATIONS: TOTAL RADIATED FIELD

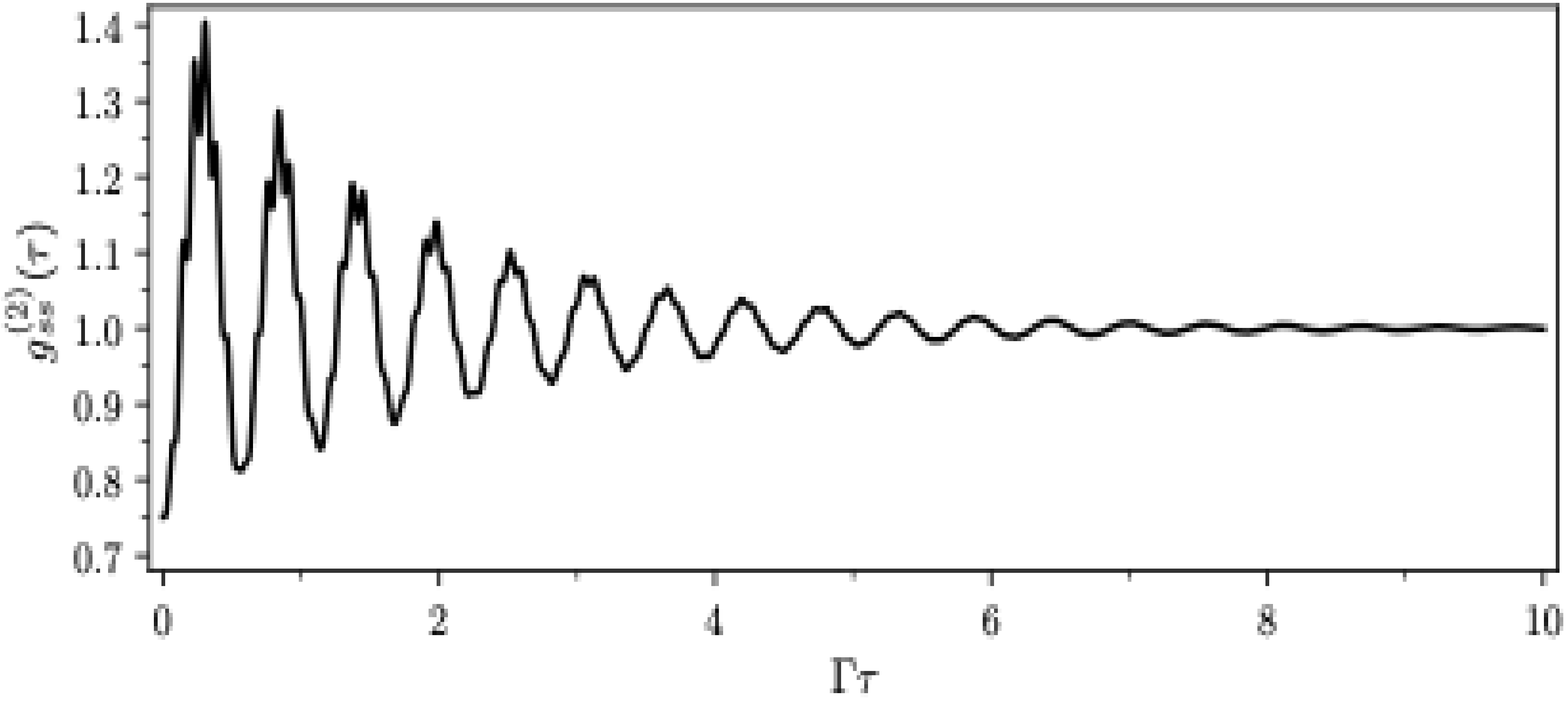
- Weak driving = strong bunching



PHOTON CORRELATIONS: TOTAL RADIATED FIELD

- Weak driving = strong bunching
- Strong driving
  - Rabi oscillations
  - Faster frequency from anharmonicity
  - Slightly antibunched

$$g^{(2)}(0) > 0.5$$



PHOTON CORRELATIONS: DRESSED-STATE APPROXIMATION

Diagonalise Hamiltonian $\boldsymbol{D} = \boldsymbol{S}^{-1} H_A \boldsymbol{S}$ $\quad \boldsymbol{D} = \begin{pmatrix} \omega_m & 0 & 0 \\ 0 & \omega_u & 0 \\ 0 & 0 & \omega_l \end{pmatrix}$ $\quad \boldsymbol{S} = [\left|m\right\rangle, \left|u\right\rangle, \left|l\right\rangle]$

Transform master equation into dressed state basis

$$\frac{\mathrm{d}\rho_D}{\mathrm{d}t} = \boldsymbol{S}^{-1} \frac{\mathrm{d}\rho}{\mathrm{d}t} \boldsymbol{S}$$

Derive second-order correlation functions for each side-peak $(\Omega \to \infty)$

$$\lambda_- = \frac{-3\xi^2\Gamma}{2\left(1+\xi^2\right)},$$

$$\lambda_+ = \frac{\Gamma\left(1-\xi^2+\xi^4\right)}{2\left(1+\xi^2\right)}.$$

$$g_0^{(2)}(\tau) = 1 + \frac{1}{2}e^{\lambda_-\tau},$$

$$g_{\pm 1}^{(2)}(\tau) = 1 - e^{\lambda_-\tau},$$

$$g_{\pm 2}^{(2)}(\tau) = 1 - e^{\lambda_-\tau},$$

$$g_{\pm 3}^{(2)}(\tau) = 1 + \frac{1}{2}e^{\lambda_-\tau} - \frac{3}{2}e^{\lambda_+\tau}.$$

PHOTON CORRELATIONS: DRESSED-STATE APPROXIMATION

Auto-correlations

$$g_0^{(2)}(\tau) = 1 + \frac{1}{2}e^{\lambda_- \tau},$$ Bunched

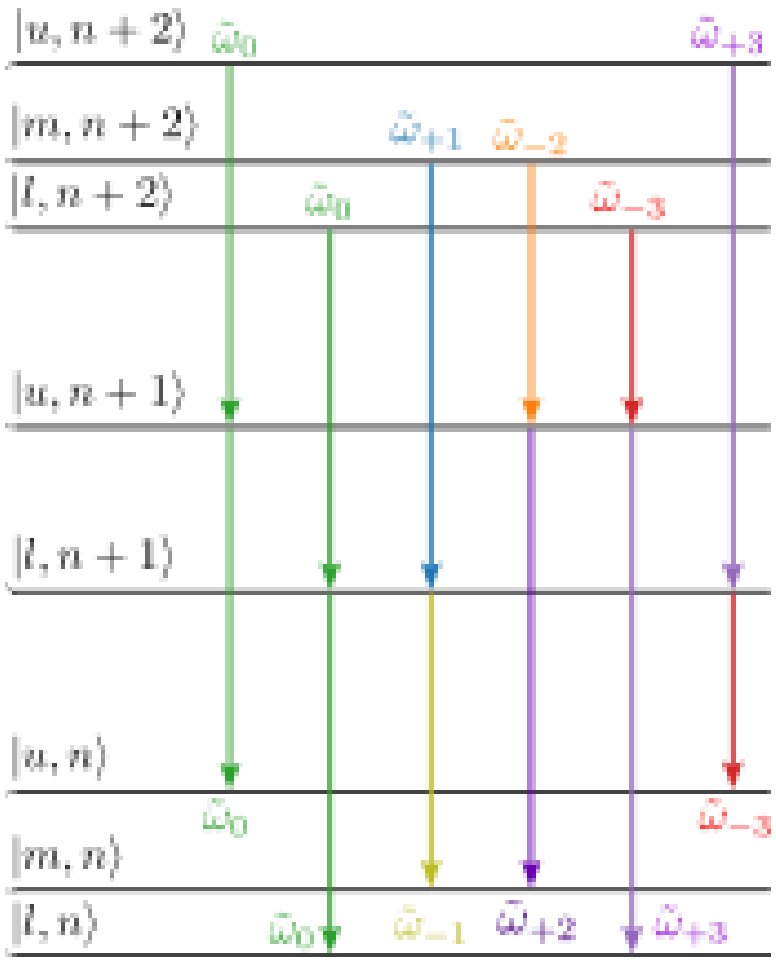
$$g_{\pm1}^{(2)}(\tau) = 1 - e^{\lambda_- \tau},$$ Anti-bunched

$$g_{\pm2}^{(2)}(\tau) = 1 - e^{\lambda_- \tau},$$ Anti-bunched

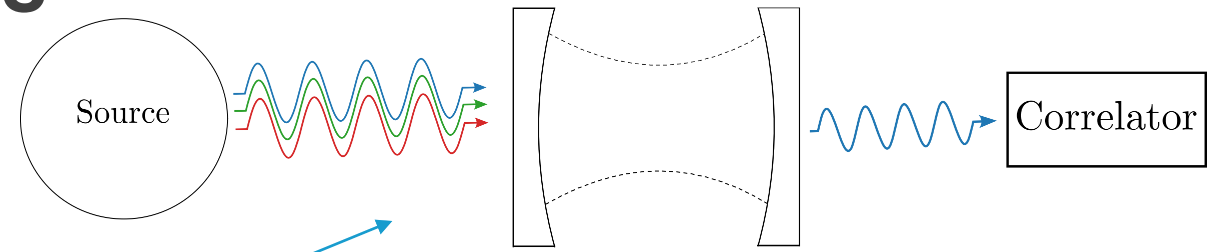
$$g_{\pm3}^{(2)}(\tau) = 1 + \frac{1}{2}e^{\lambda_- \tau} - \frac{3}{2}e^{\lambda_+ \tau}.$$ Anti-bunched

Cross-correlations: sidepeak-to-sidepeak

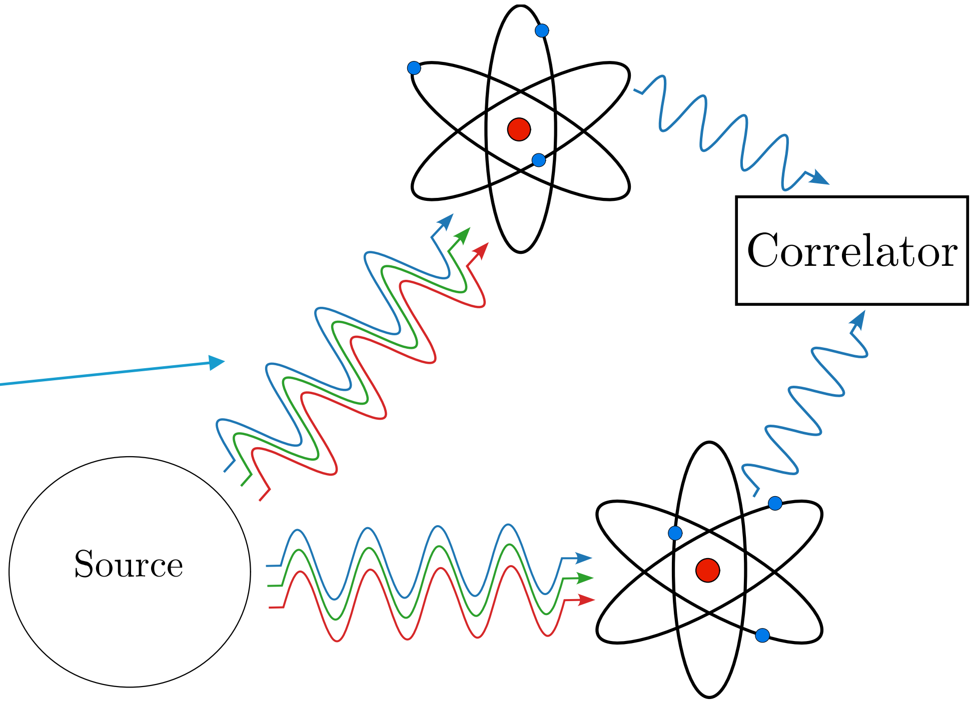
$$g_{+i,-i}^{(2)}(\tau) = 1 + \frac{1}{2}e^{\lambda_- \tau} + \frac{3}{2}e^{\lambda_+ \tau}$$ Correlated



FREQUENCY-FILTERING METHODS



- Perturbation method
  - Holdaway et al., PRA **98**, 063828 (2018).
- Superoperator decomposition
  - Kamide et al., PRA **92**, 033833 (2015).
- Fabry-Pérot interferometers
  - *Phys. Rev. A* **42**, 503 (1990); *Phys .Rev. Lett* **125**, 043603 (2020); *Phys .Rev. Lett* **125**, 170402 (2020);
- Detector atoms
  - del Valle et al., PRL **109**, 183601 (2012).



THE (QUANTUM) FABRY-PÉROT INTERFEROMETER

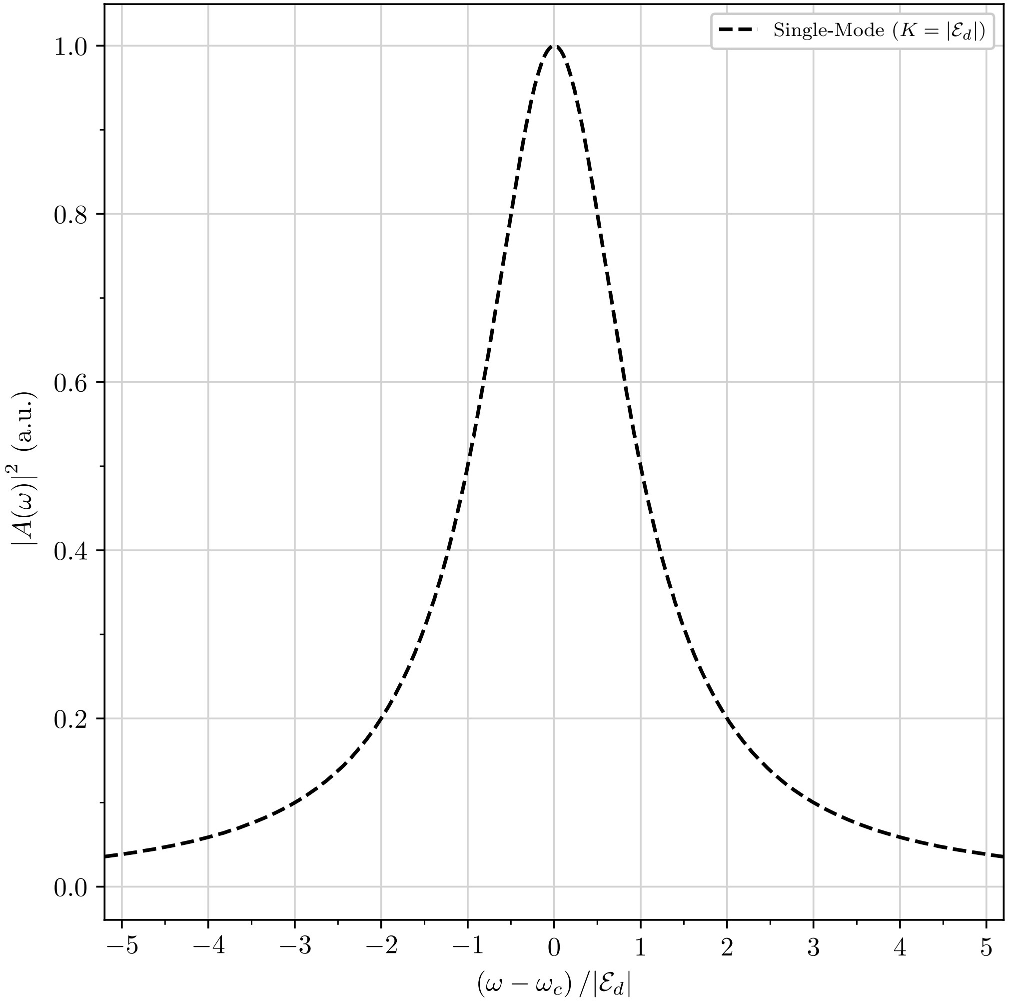
- Frequency response is Lorentzian

Driving amplitude

$$|A(\omega)|^2 = \frac{|\mathcal{E}_d|^2}{K^2 + (\omega - \omega_c)^2}$$

bandwidth

Resonance frequency



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- Frequency response is Lorentzian

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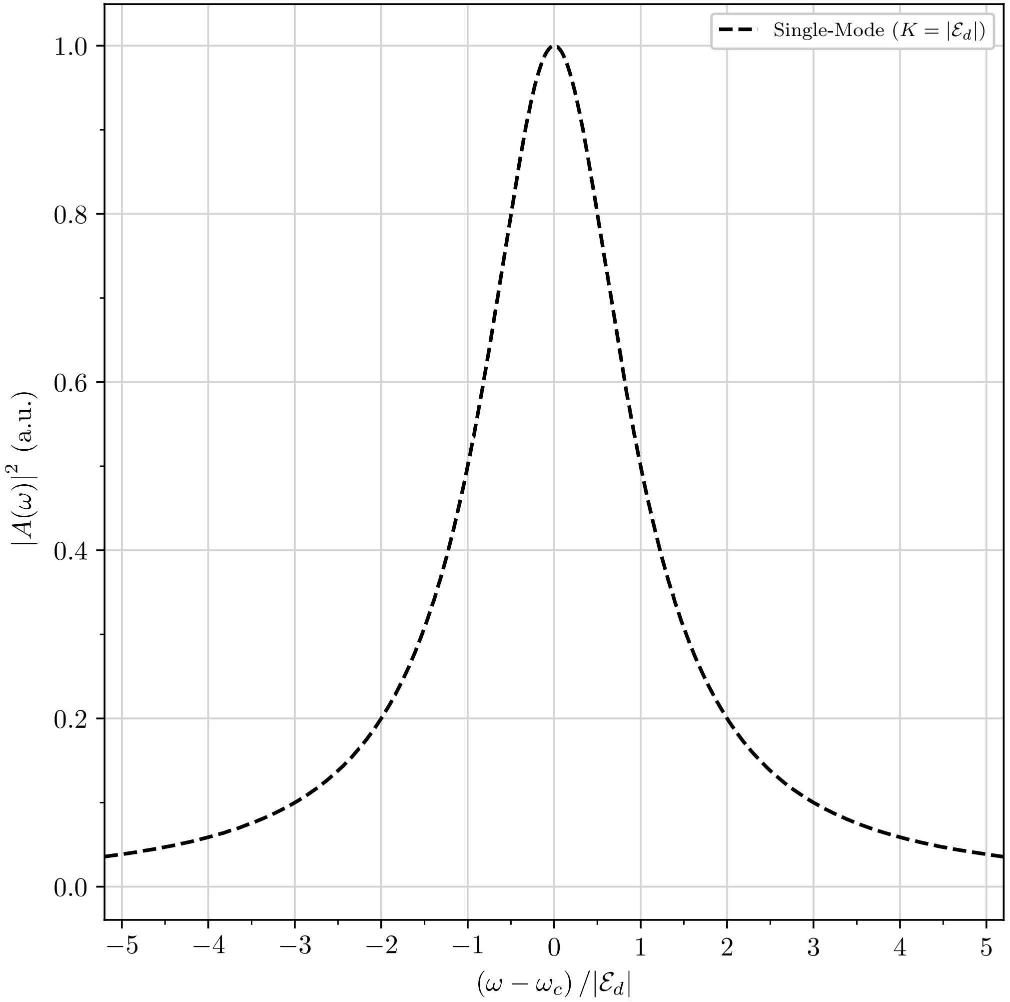
$$|A(\omega)|^2 = \frac{|\mathcal{E}_d|^2}{K^2 + (\omega - \omega_c)^2}$$

bandwidth

Resonance frequency

- Trade-off between frequency temporal response and filter bandwidth K
  - Fast temporal response = wide frequency distribution
  - Narrow frequency distribution = slow temporal response

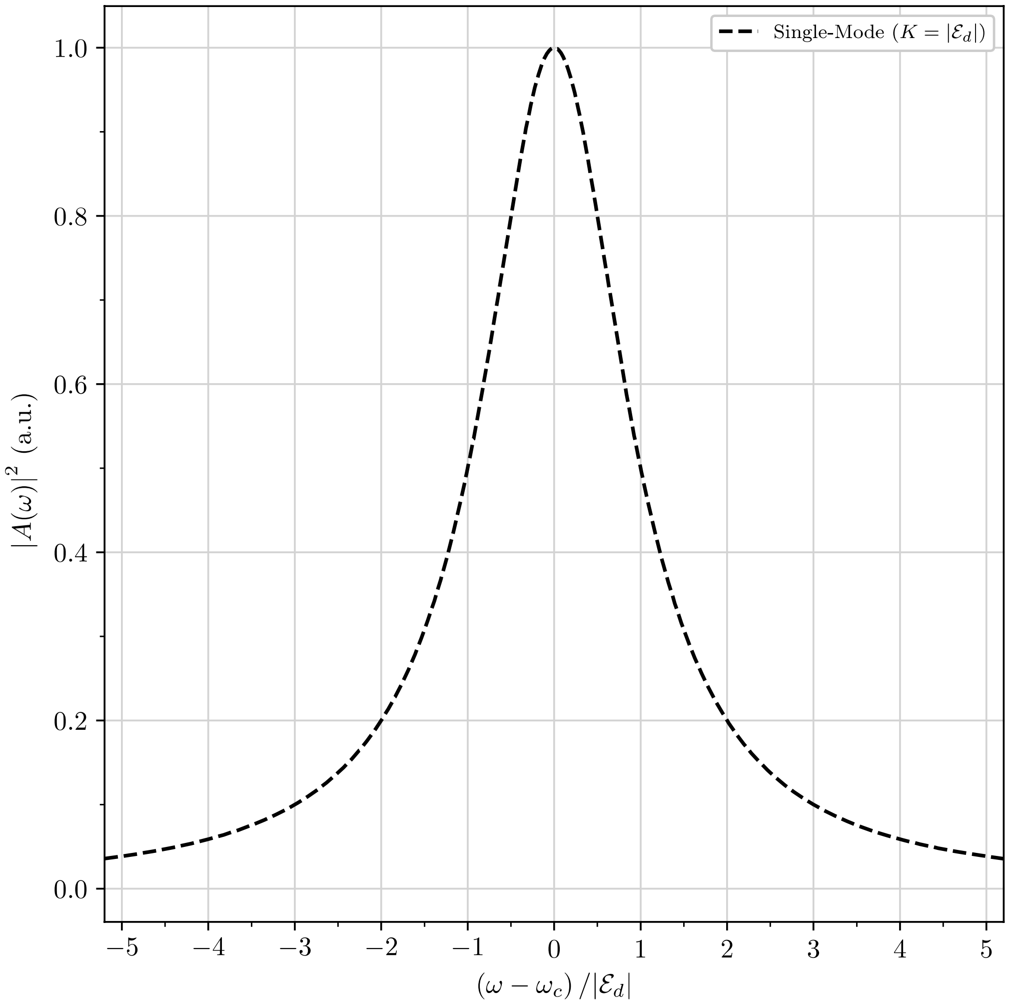
- Lorentzian distribution has wide-reaching “tails”, with large frequency response



THE PROBLEM

- Want to measure photon correlations *as emitted* by a quantum system
  - Need a *fast* temporal response
  - But still want near perfect frequency isolation

- Wide Lorentzian tails give poor frequency isolation

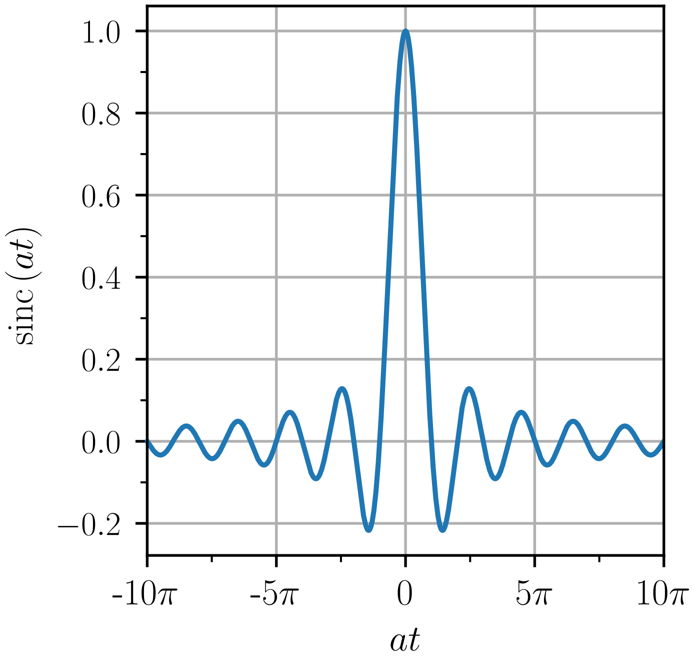


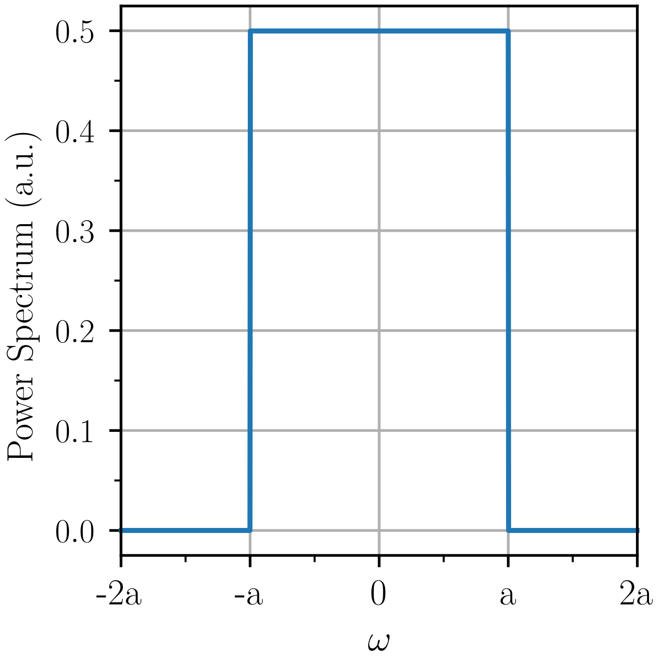
THE “IDEAL” BOX FILTER

- Fourier transform a *rectangular function* is a *“sinc” function*

$$\mathrm{sinc}(at) = \frac{\sin\left(at\right)}{at}$$

- A complete sinc function (in positive and negative time) is unphysical / *“non-causal”*





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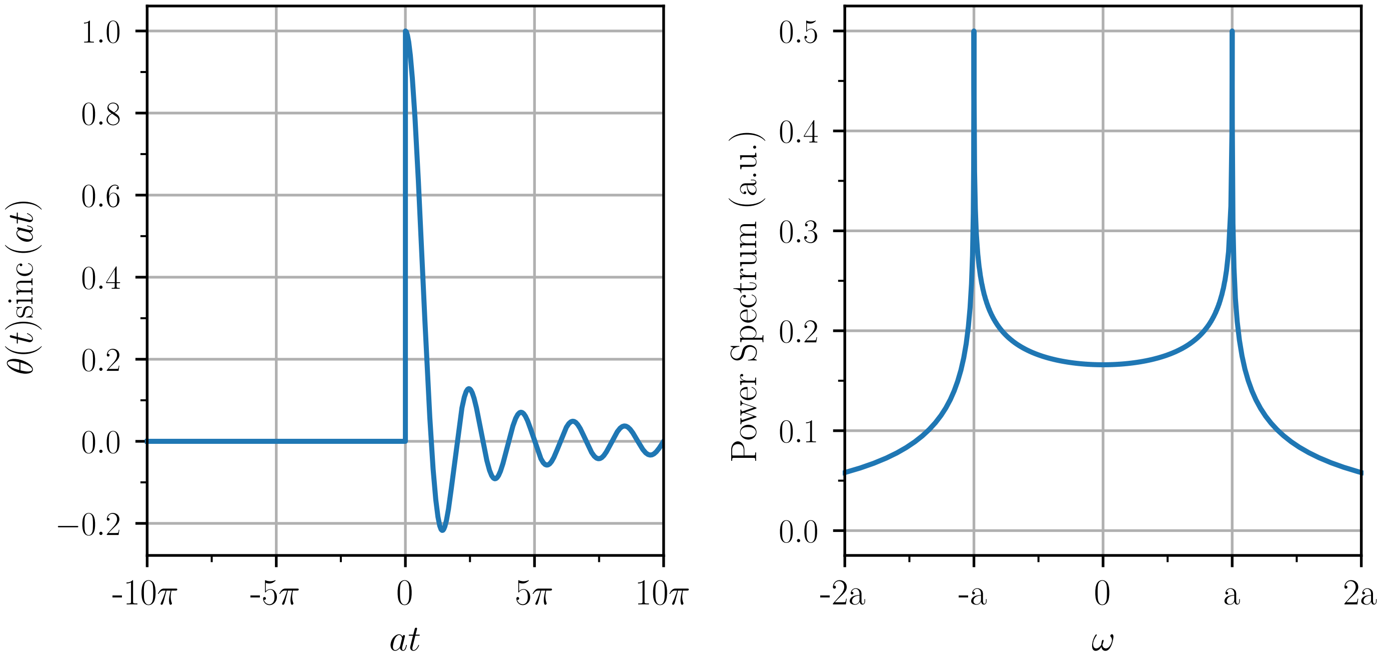
- Taking only “positive” time results in wide, ”Batman-like” response.



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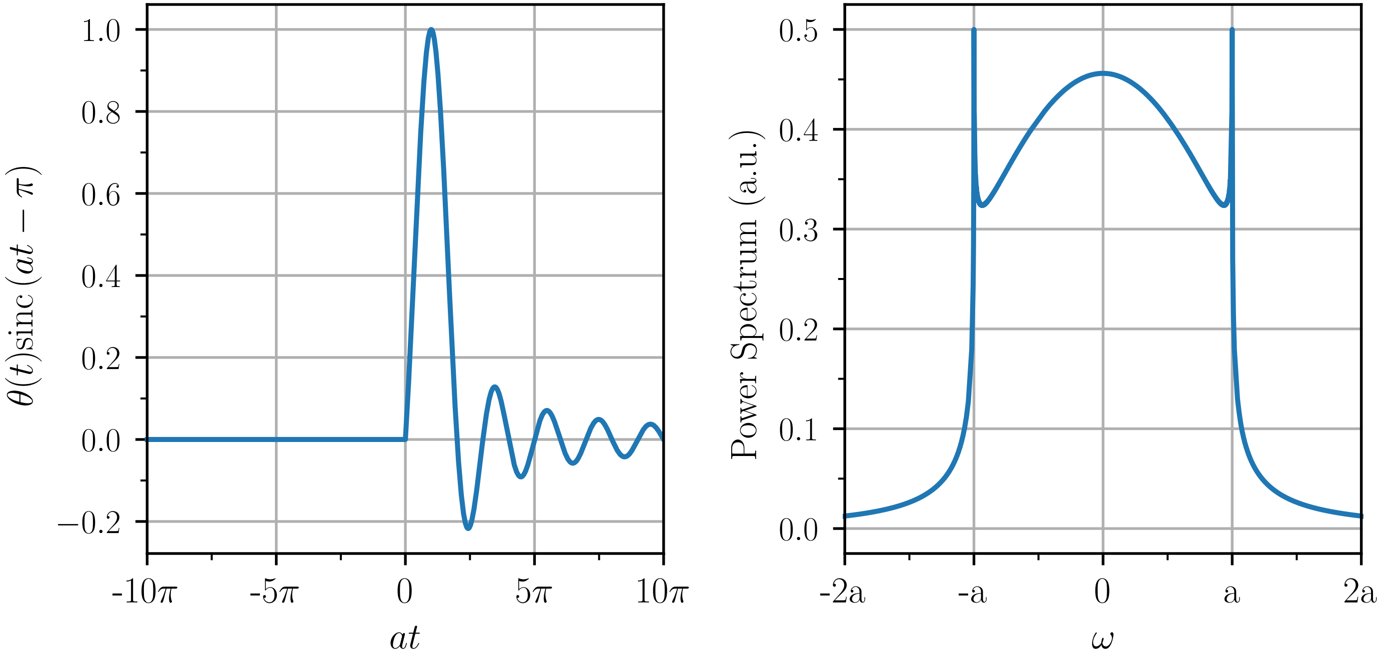
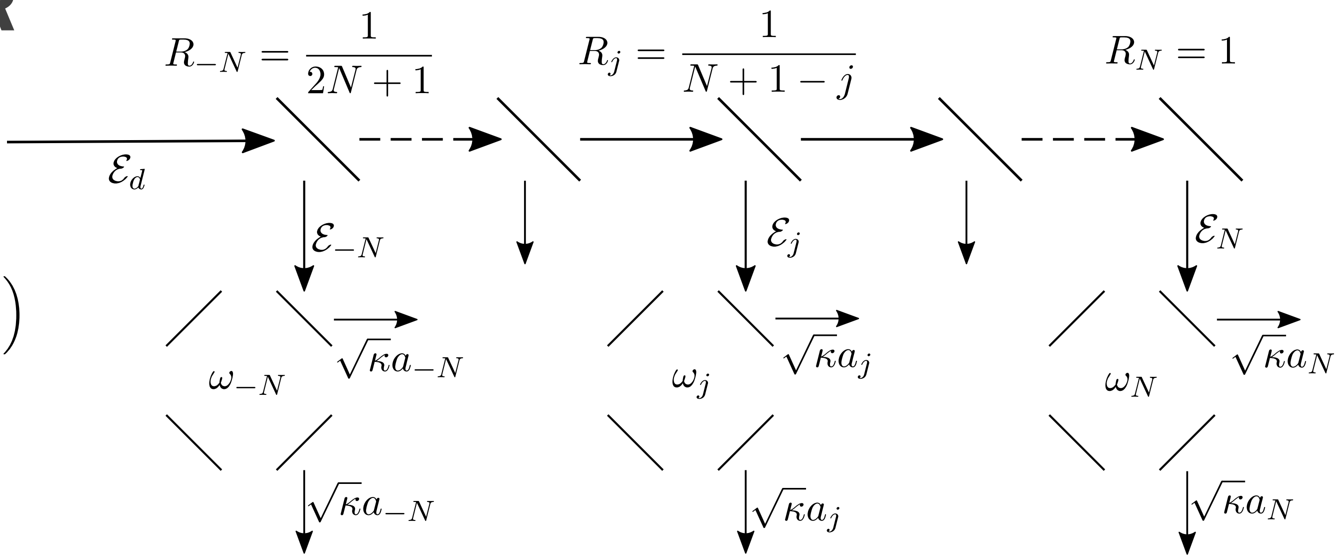
- “Phase modulation” recovers sharp edges



THE MULTI-MODE ARRAY FILTER



- Extend simple Hamiltonian

$$H = \hbar \sum_{j=-N}^{N} \omega_j a_j^\dagger a_j + i\hbar \sum_{j=-N}^{N} \left(\mathcal{E}_j e^{-i\omega t} a_j^\dagger - \mathcal{E}_j^* e^{i\omega t} a_j \right)$$

- Total number of filter modes – $2N+1$
- j^{th} mode resonance frequency – $\omega_j = \omega_0 + j\delta\omega$
- Frequency separation – $\delta\omega$
- Mode-dependent phase modulation – $\mathcal{E}_j = \frac{\mathcal{E}_d}{\sqrt{2N+1}} e^{ij\pi/N}$
- Combined/collective output – $A = \frac{1}{\sqrt{2N+1}} \sum_{j=-N}^{N} a_j$

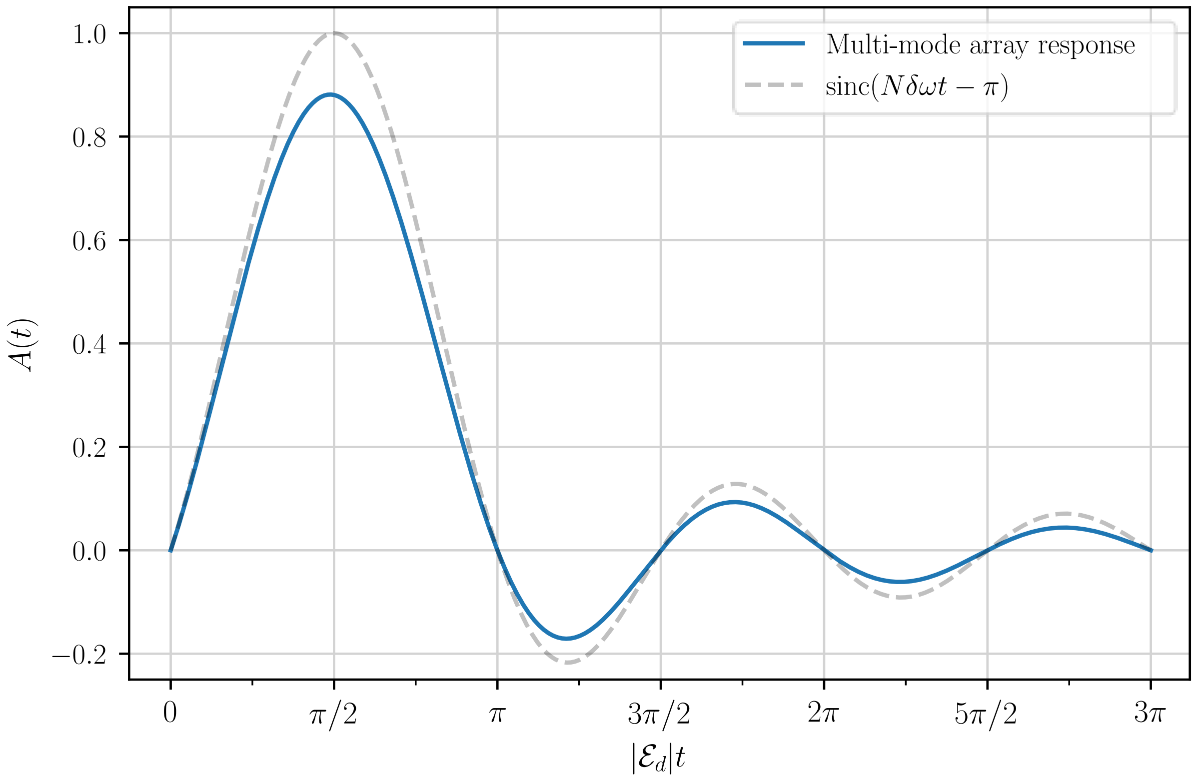
MULTI-MODE ARRAY – TEMPORAL RESPONSE

- Consider impulse driving

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_j = -\left(\kappa + i\omega_j\right)\alpha_j + \mathcal{E}_j\delta(t)$$

- Multi-mode array temporal response is a shifted sinc-function

$$A(t) = \sum_{j=-N}^{N}\frac{\alpha_j(t)}{\sqrt{2N+1}} \sim \frac{2N\mathcal{E}_d}{2N+1}\theta(t)e^{-(\kappa+i\omega_0)t}\mathrm{sinc}\left(N\delta\omega t - \pi\right)$$



Parameters: $N = 10, K = N\delta\omega = 2, \kappa = 2.5\delta\omega$

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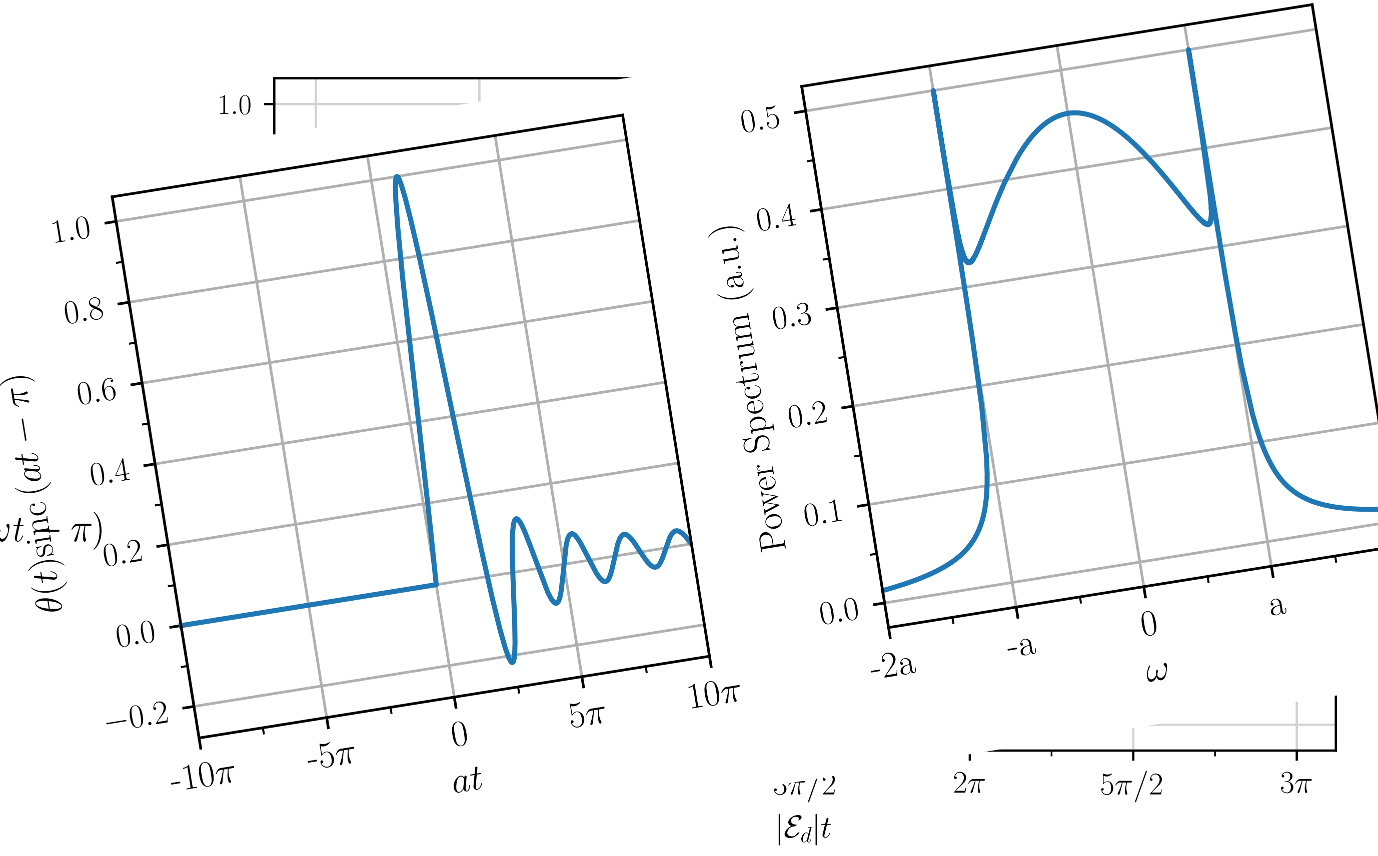


Parameters: $N = 10, K = N\delta\omega = 2, \kappa = 0.25\delta\omega$

Phys. Rev. A **100**, 023719 (2024)

MULTI-MODE ARRAY – FREQUENCY RESPONSE

- Consider continuous driving

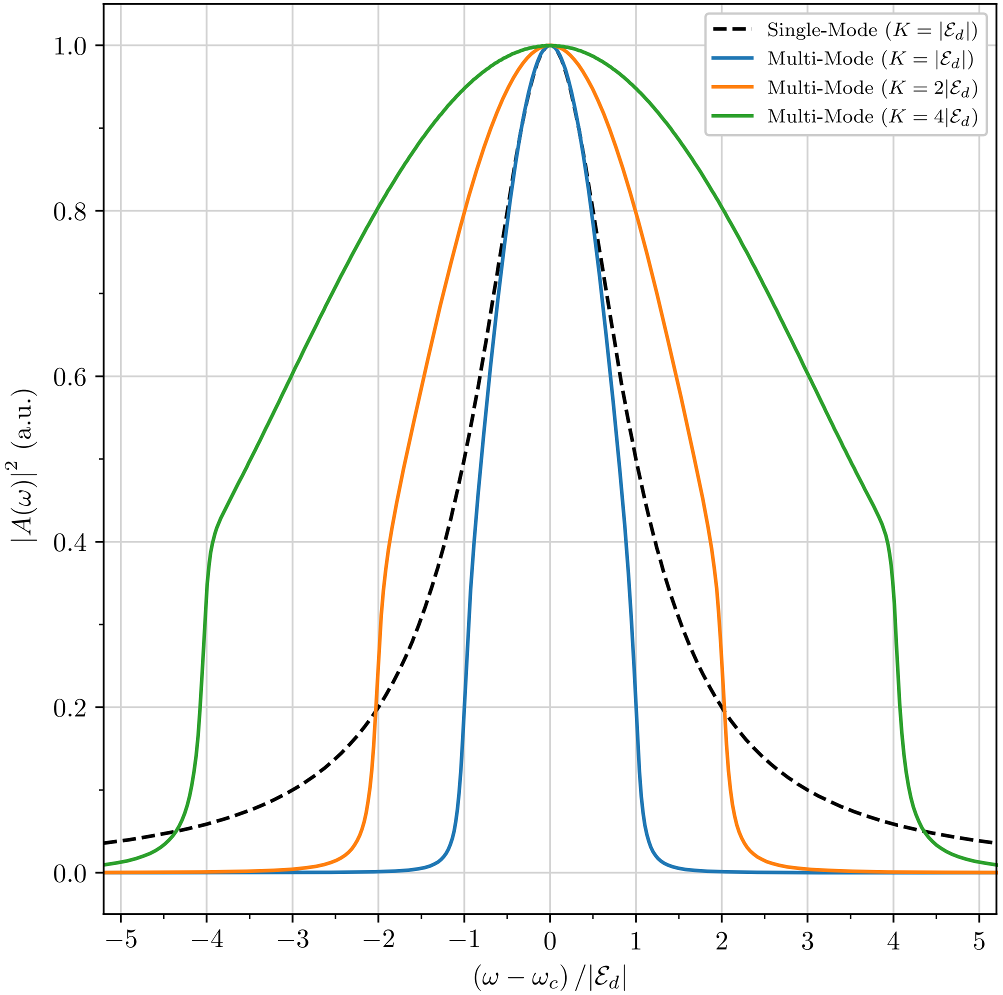
$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_j = -\left(\kappa + i\omega_j\right)\alpha_j + \mathcal{E}_j e^{-i\omega t}, \quad A(\omega) = \sum_{j=-N}^{N} \frac{\alpha_j(\omega)}{\sqrt{2N+1}}$$

- Multi-mode array temporal response is a shifted sinc-function

- Filter halfwidth *K*
  - Single-mode filter – $K = \kappa$
  - Multi-mode array filter – $K = N\delta\omega$

- Sharper frequency response cut-off

- Larger bandwidth = faster (better) temporal response

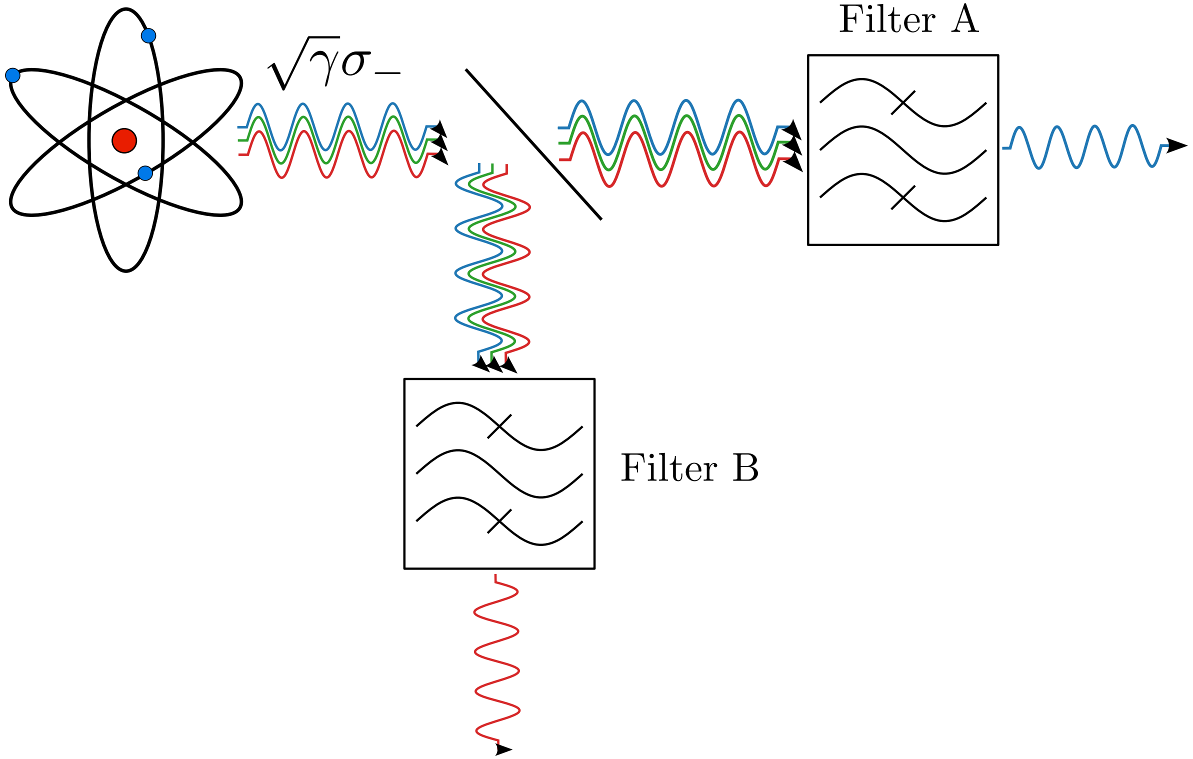


Parameters: $N = 80, \kappa = 0.25\delta\omega$

THE MULTI-MODE ARRAY FILTER

Master equation

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} = {} & \frac{1}{i\hbar}[H_A,\rho] + \frac{\Gamma}{2}\Lambda(\Sigma_-)\rho \\
& - i\sum_{j=-N}^{N}\Delta\omega_j^{(a)} a_j^\dagger a_j + \frac{\kappa}{2}\sum_{j=-N}^{N}\Lambda(a_j)\rho \\
& - \sum_{j=-N}^{N}\mathcal{E}_j\left(a_j^\dagger\Sigma_-\rho - \Sigma_-\rho a_j^\dagger\right) - \sum_{j=-N}^{N}\mathcal{E}_j^*\left(\rho\Sigma_+ a_j - a_j\rho\Sigma_+\right) \\
& - i\sum_{j=-N}^{N}\Delta\omega_j^{(b)} b_j^\dagger b_j + \frac{\kappa}{2}\sum_{j=-N}^{N}\Lambda(b_j)\rho \\
& - \sum_{j=-N}^{N}\mathcal{E}_j\left(b_j^\dagger\Sigma_-\rho - \Sigma_-\rho b_j^\dagger\right) - \sum_{j=-N}^{N}\mathcal{E}_j^*\left(\rho\Sigma_+ b_j - b_j\rho\Sigma_+\right)
\end{aligned}
$$



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Master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \boxed{\frac{1}{i\hbar}[H_A,\rho] + \frac{\Gamma}{2}\Lambda(\Sigma_-)\rho} \quad \text{Driven atom}$$

$$- i \sum_{j=-N}^{N} \Delta\omega_j^{(a)} a_j^\dagger a_j + \frac{\kappa}{2}\sum_{j=-N}^{N} \Lambda(a_j)\rho$$

$$- \sum_{j=-N}^{N} \mathcal{E}_j \left(a_j^\dagger \Sigma_- \rho - \Sigma_- \rho a_j^\dagger\right) - \sum_{j=-N}^{N} \mathcal{E}_j^* \left(\rho \Sigma_+ a_j - a_j \rho \Sigma_+\right)$$

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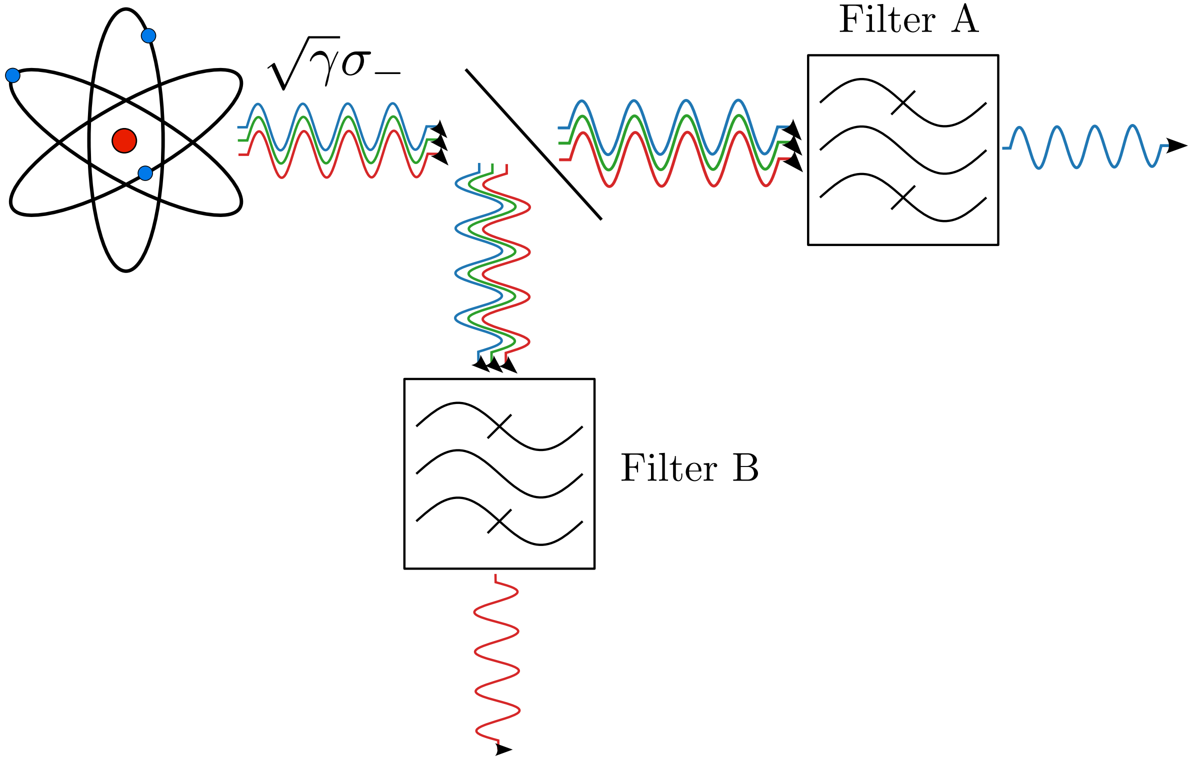

THE MULTI-MODE ARRAY FILTER

Master equation

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} = {} & \boxed{\frac{1}{i\hbar}[H_A,\rho] + \frac{\Gamma}{2}\Lambda(\Sigma_-)\rho} \quad \text{Driven atom} \\ & \boxed{- i \sum_{j=-N}^{N} \Delta\omega_j^{(a)} a_j^\dagger a_j + \frac{\kappa}{2} \sum_{j=-N}^{N} \Lambda(a_j)\rho} \quad \text{Array Filter A} \\ & - \sum_{j=-N}^{N} \mathcal{E}_j \left(a_j^\dagger \Sigma_- \rho - \Sigma_- \rho a_j^\dagger \right) - \sum_{j=-N}^{N} \mathcal{E}_j^* \left(\rho \Sigma_+ a_j - a_j \rho \Sigma_+ \right) \\ & \boxed{- i \sum_{j=-N}^{N} \Delta\omega_j^{(b)} b_j^\dagger b_j + \frac{\kappa}{2} \sum_{j=-N}^{N} \Lambda(b_j)\rho} \quad \text{Array Filter B} \\ & - \sum_{j=-N}^{N} \mathcal{E}_j \left(b_j^\dagger \Sigma_- \rho - \Sigma_- \rho b_j^\dagger \right) - \sum_{j=-N}^{N} \mathcal{E}_j^* \left(\rho \Sigma_+ b_j - b_j \rho \Sigma_+ \right)\end{aligned}$$

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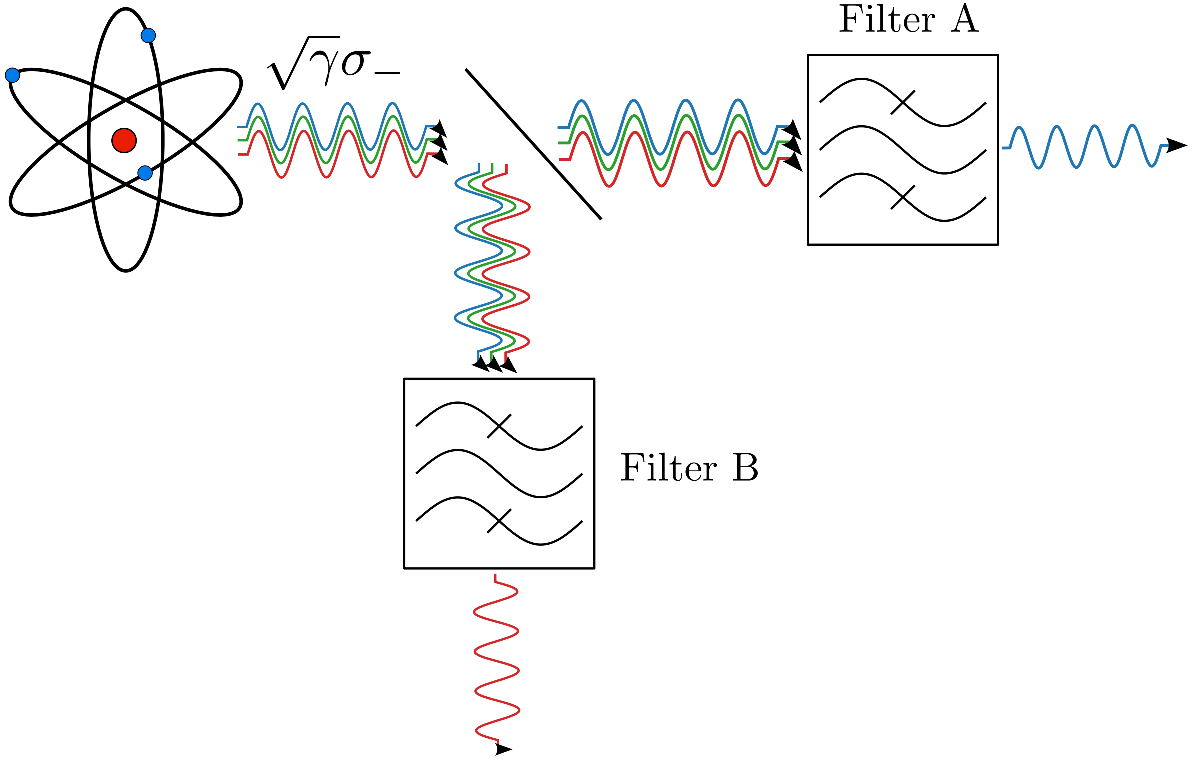

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Master equation

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Driven atom

$$- i \sum_{j=-N}^{N} \Delta\omega_j^{(a)} a_j^\dagger a_j + \frac{\kappa}{2} \sum_{j=-N}^{N} \Lambda(a_j)\rho$$

Array Filter A

$$- \sum_{j=-N}^{N} \mathcal{E}_j \left(a_j^\dagger \Sigma_- \rho - \Sigma_- \rho a_j^\dagger \right) - \sum_{j=-N}^{N} \mathcal{E}_j^* \left(\rho \Sigma_+ a_j - a_j \rho \Sigma_+ \right)$$

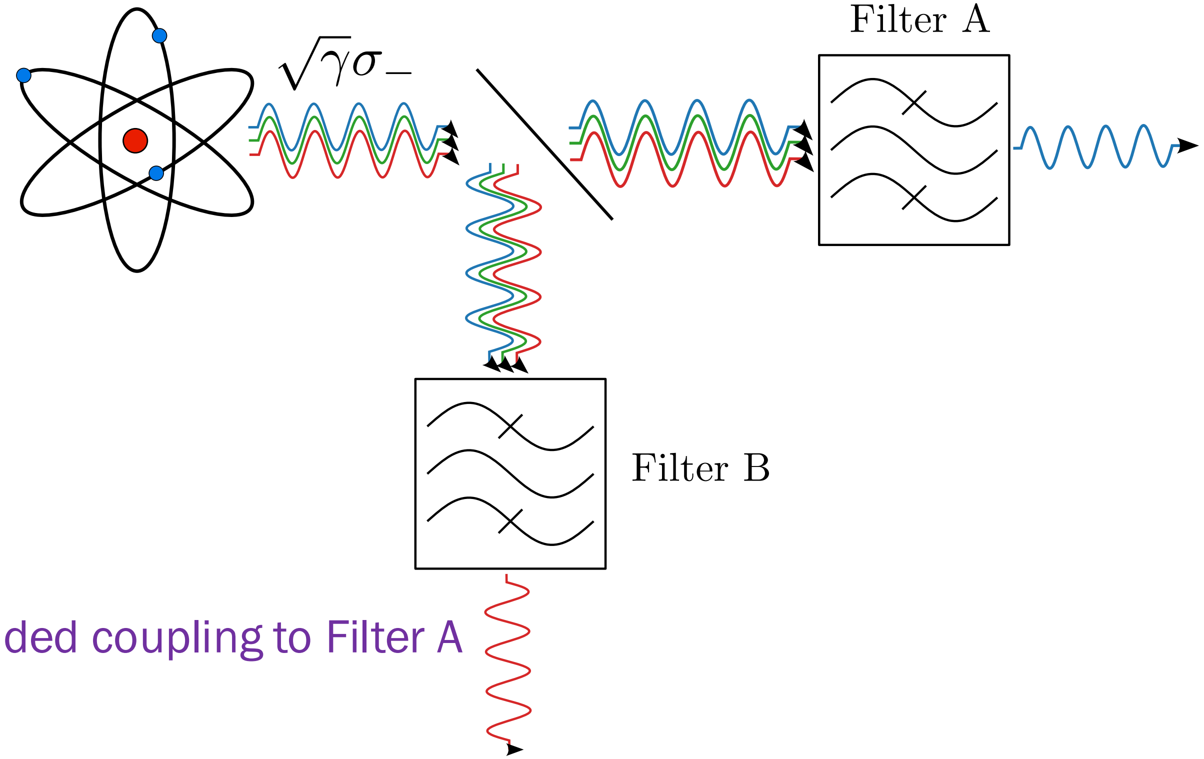
Cascaded coupling to Filter A

$$- i \sum_{j=-N}^{N} \Delta\omega_j^{(b)} b_j^\dagger b_j + \frac{\kappa}{2} \sum_{j=-N}^{N} \Lambda(b_j)\rho$$

Array Filter B

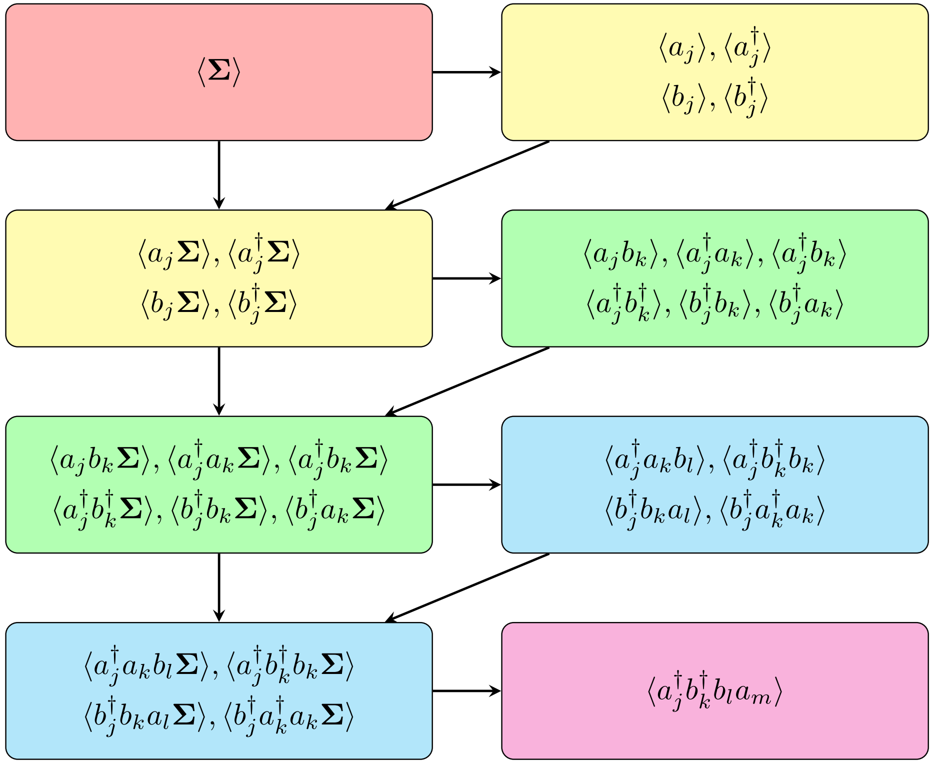
$$- \sum_{j=-N}^{N} \mathcal{E}_j \left(b_j^\dagger \Sigma_- \rho - \Sigma_- \rho b_j^\dagger \right) - \sum_{j=-N}^{N} \mathcal{E}_j^* \left(\rho \Sigma_+ b_j - b_j \rho \Sigma_+ \right)$$

Cascaded Coupling to Filter B



$$\Lambda(X)\bullet = 2X \bullet X^\dagger - X^\dagger X \bullet - \bullet X^\dagger X$$

CALCULATION METHOD – THE MOMENT EQUATIONS



$$\langle \boldsymbol{\Sigma} \rangle$$

$$\langle a_j \rangle, \langle a_j^\dagger \rangle$$
$$\langle b_j \rangle, \langle b_j^\dagger \rangle$$

$$\langle a_j \boldsymbol{\Sigma} \rangle, \langle a_j^\dagger \boldsymbol{\Sigma} \rangle$$
$$\langle b_j \boldsymbol{\Sigma} \rangle, \langle b_j^\dagger \boldsymbol{\Sigma} \rangle$$

$$\langle a_j b_k \rangle, \langle a_j^\dagger a_k \rangle, \langle a_j^\dagger b_k \rangle$$
$$\langle a_j^\dagger b_k^\dagger \rangle, \langle b_j^\dagger b_k \rangle, \langle b_j^\dagger a_k \rangle$$

$$\langle a_j b_k \boldsymbol{\Sigma} \rangle, \langle a_j^\dagger a_k \boldsymbol{\Sigma} \rangle, \langle a_j^\dagger b_k \boldsymbol{\Sigma} \rangle$$
$$\langle a_j^\dagger b_k^\dagger \boldsymbol{\Sigma} \rangle, \langle b_j^\dagger b_k \boldsymbol{\Sigma} \rangle, \langle b_j^\dagger a_k \boldsymbol{\Sigma} \rangle$$

$$\langle a_j^\dagger a_k b_l \rangle, \langle a_j^\dagger b_k^\dagger b_k \rangle$$
$$\langle b_j^\dagger b_k a_l \rangle, \langle b_j^\dagger a_k^\dagger a_k \rangle$$

$$\langle a_j^\dagger a_k b_l \boldsymbol{\Sigma} \rangle, \langle a_j^\dagger b_k^\dagger b_k \boldsymbol{\Sigma} \rangle$$
$$\langle b_j^\dagger b_k a_l \boldsymbol{\Sigma} \rangle, \langle b_j^\dagger a_k^\dagger a_k \boldsymbol{\Sigma} \rangle$$

$$\langle a_j^\dagger b_k^\dagger b_l a_m \rangle$$

- Frequency-filtered second-order correlation function:

$$g^{(2)}(\alpha, 0; \beta, \tau) = \frac{\langle A^\dagger(0) B^\dagger B(\tau) A(0) \rangle_{ss}}{\langle A^\dagger A \rangle_{ss} \langle B^\dagger B \rangle_{ss}}$$

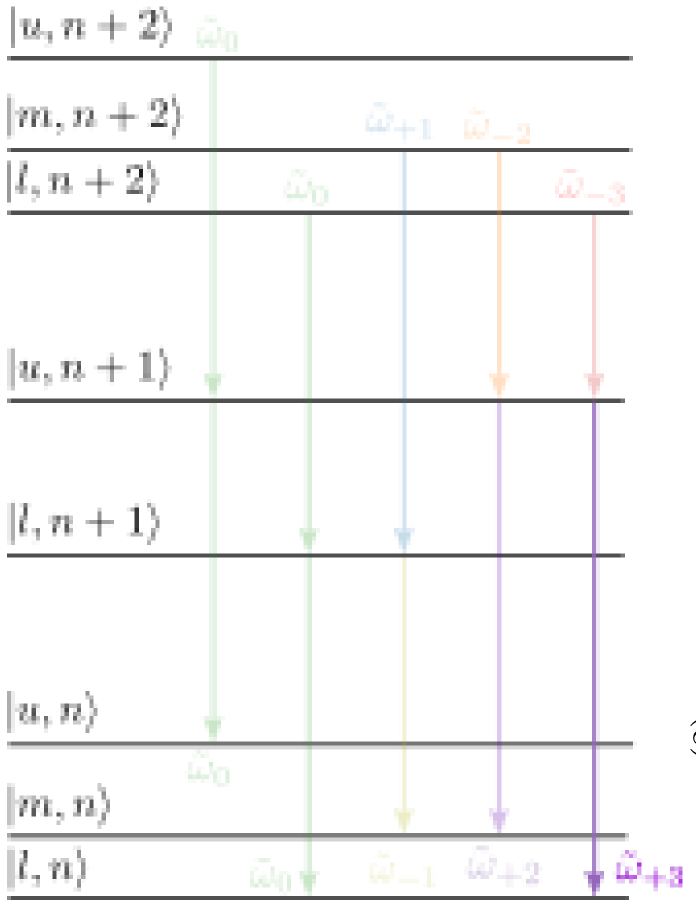
  - Resonance frequency of filter *A* – α
  - Resonance frequency of filter *B* – β

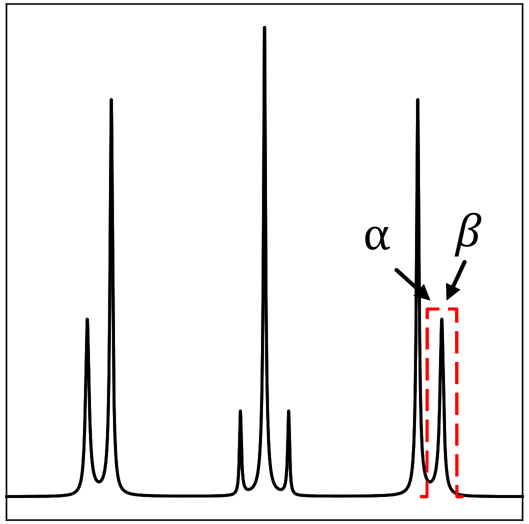
- Collective mode annihilation operators

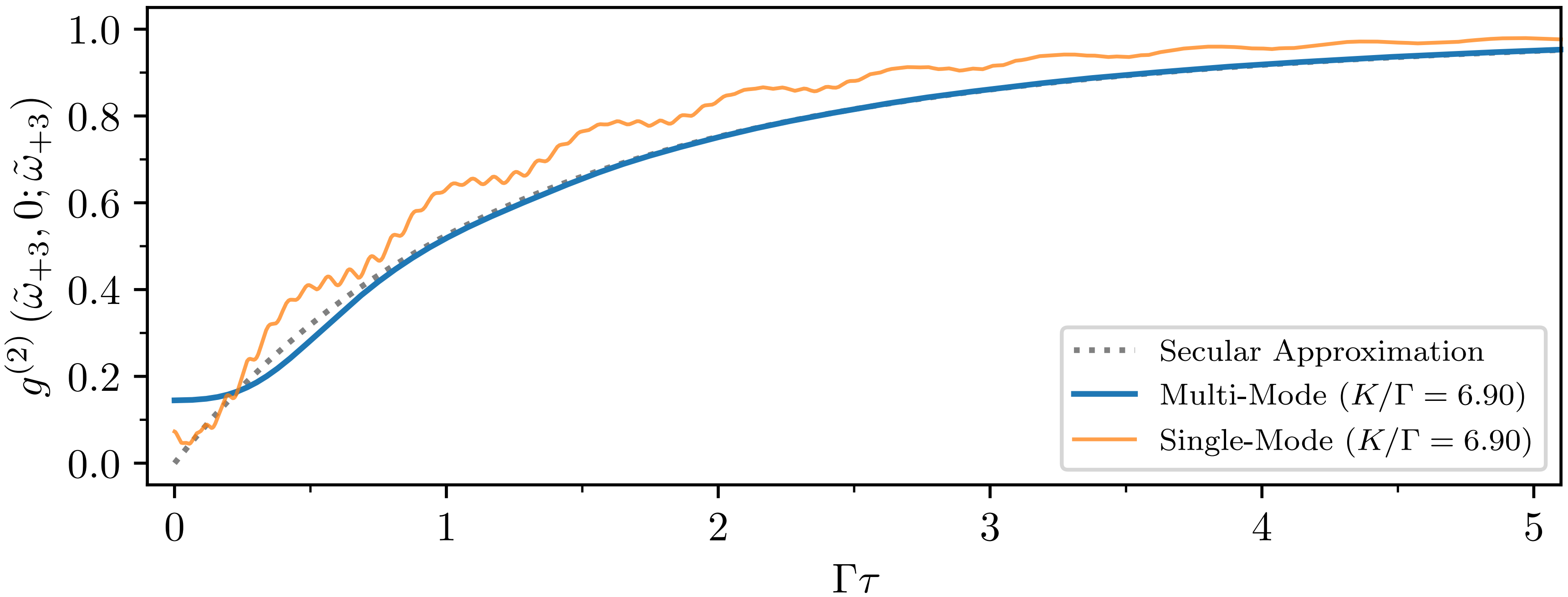
$$A = \sum_{j=-N}^{N} a_j, \quad B = \sum_{j=-N}^{N} b_j$$

- Moment equations – efficient method for calculating
  - (*N*=20) ~20 hours for master equation method
  - (*N*=100) ~1/2 second for moment equation method

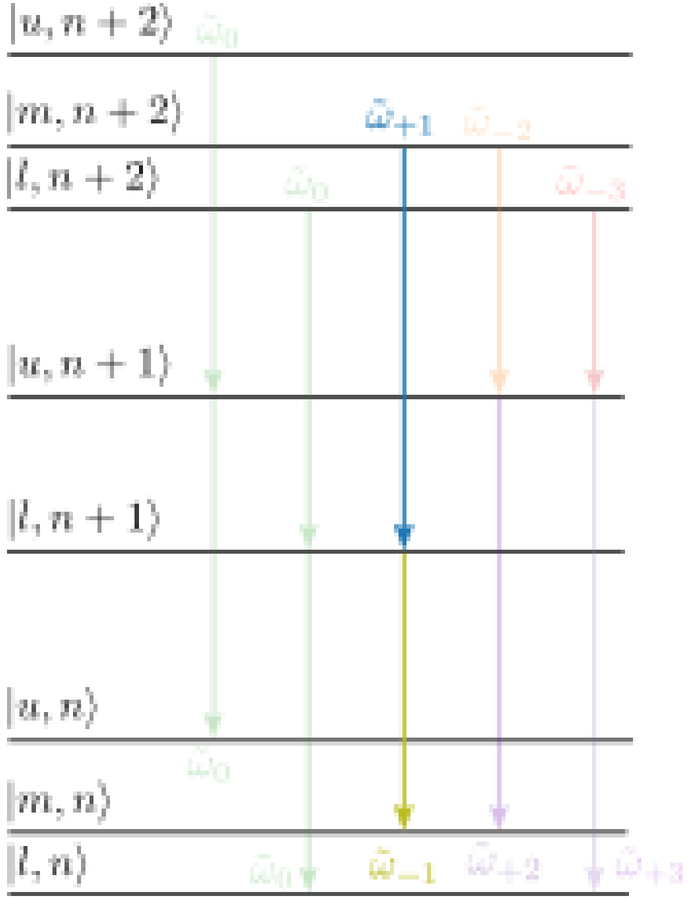
FREQUENCY-FILTERED AUTO-CORRELATIONS

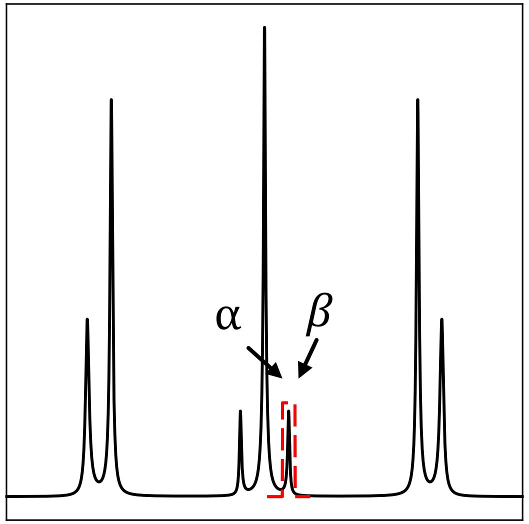


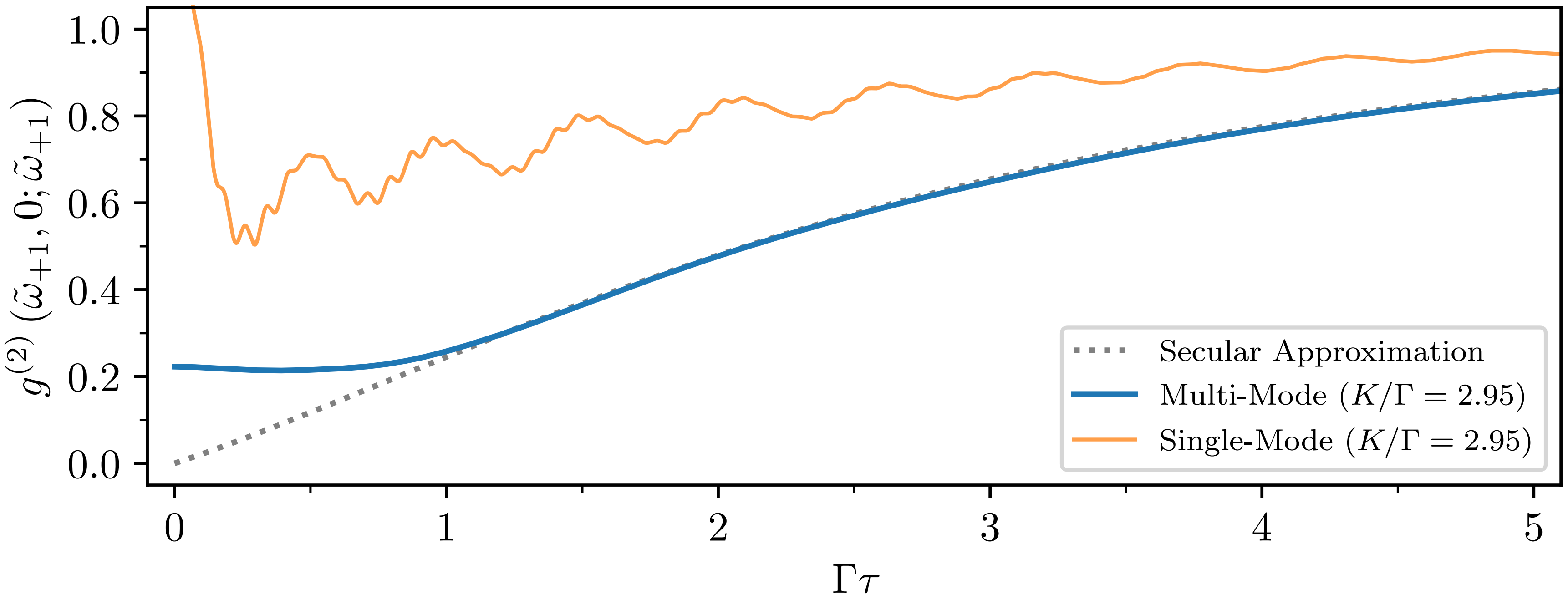




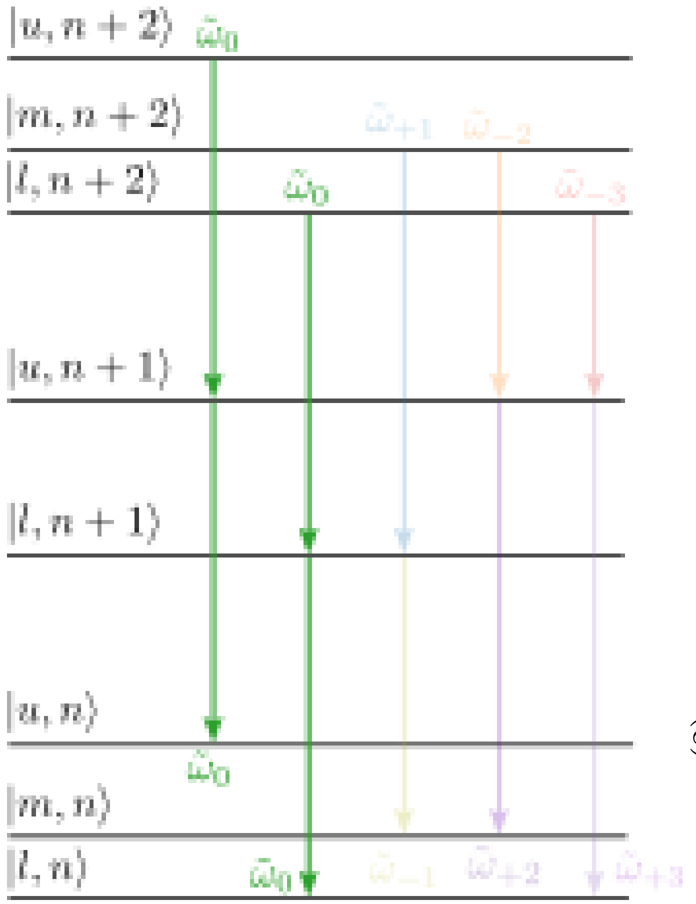
FREQUENCY-FILTERED AUTO-CORRELATIONS

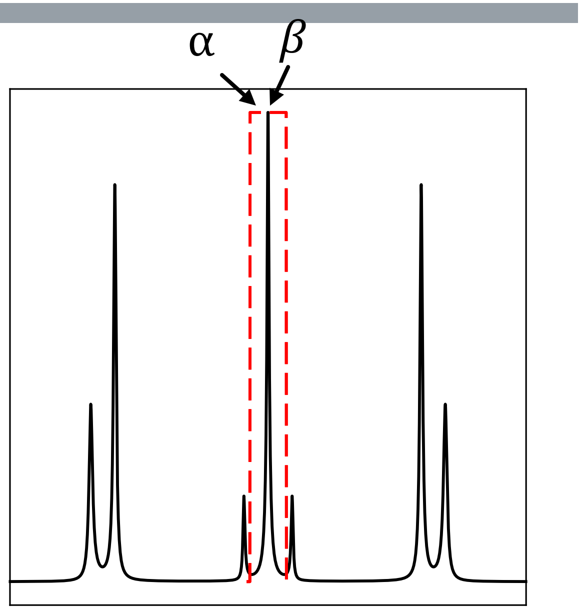


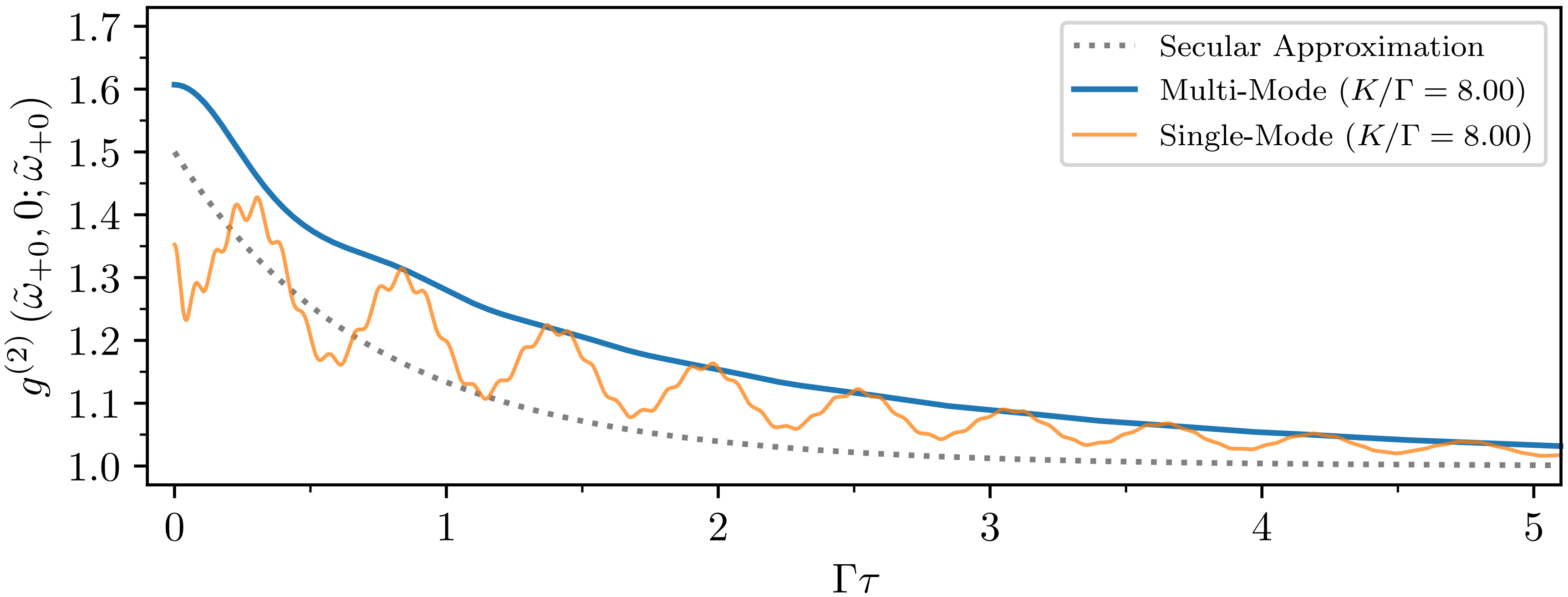




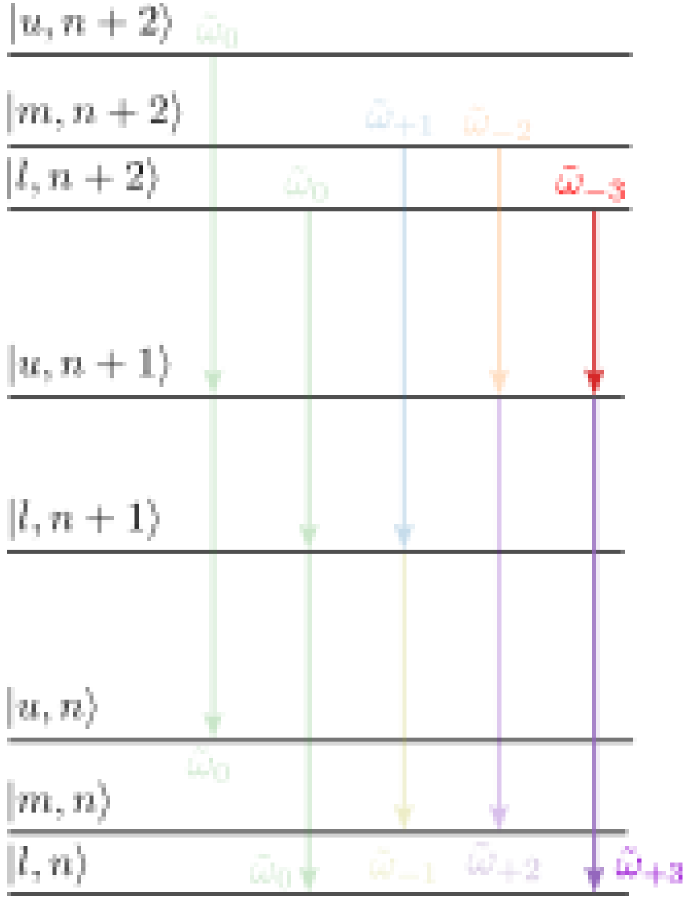
FREQUENCY-FILTERED AUTO-CORRELATIONS



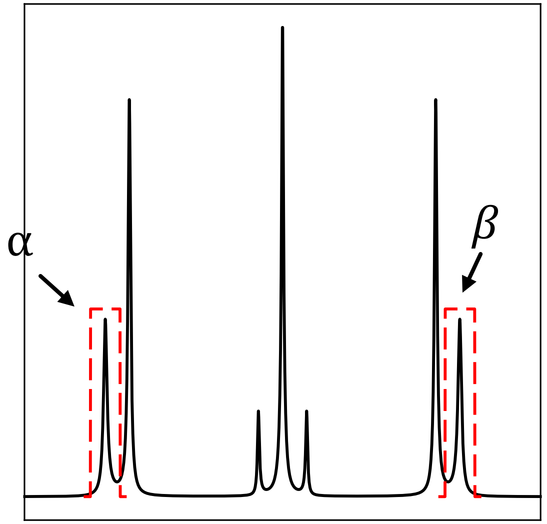


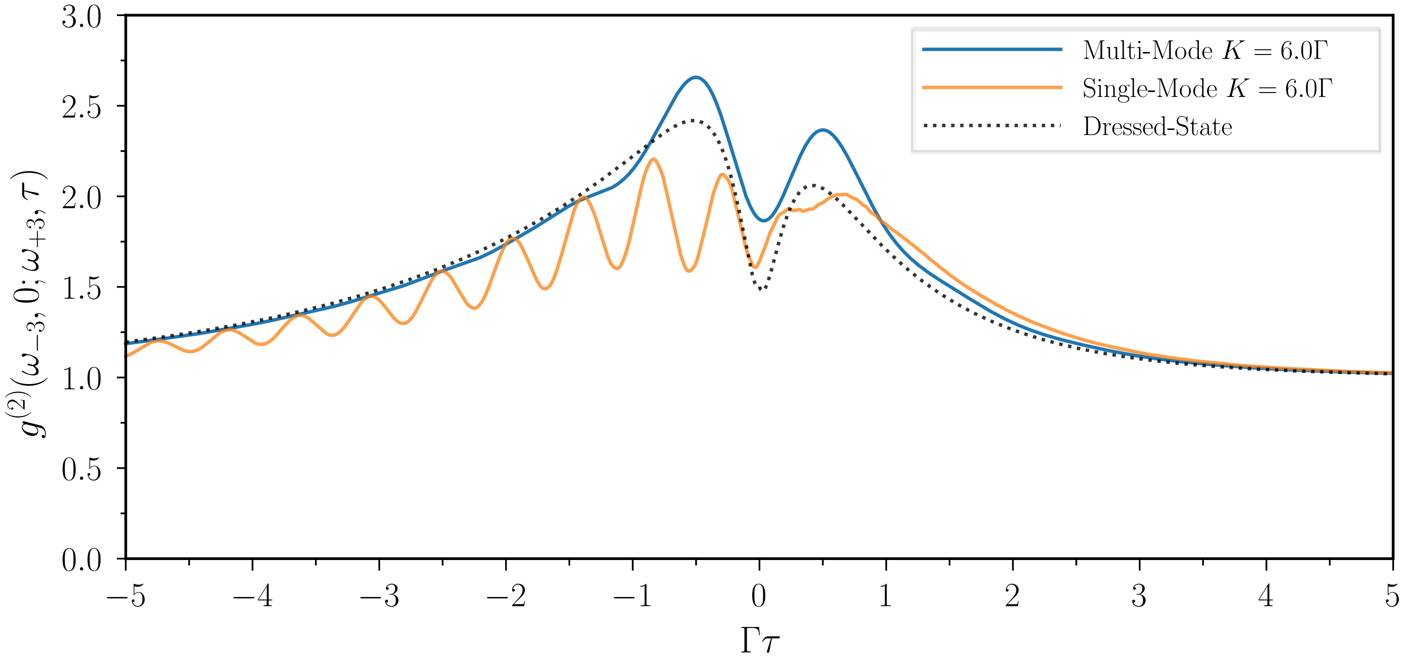


FREQUENCY-FILTERED AUTO-CORRELATIONS

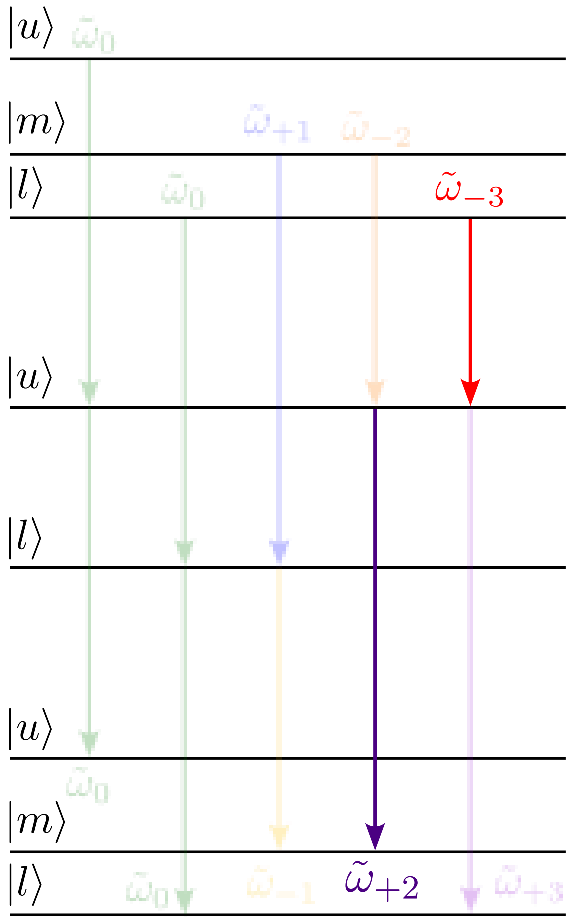


Interference of time-ordering
PRL **67**, 2443 (1991)
PRA **45**, 8045 (1992)

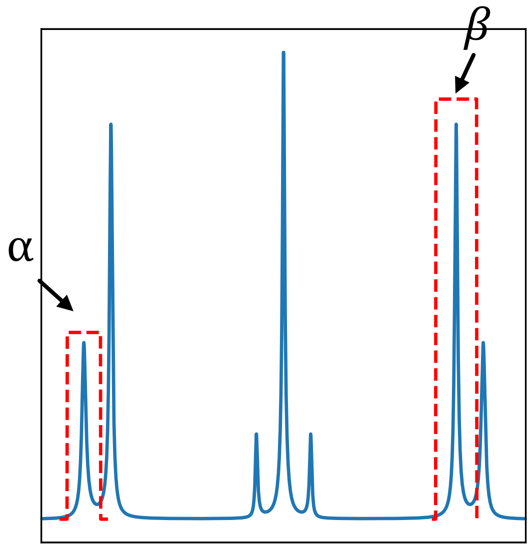


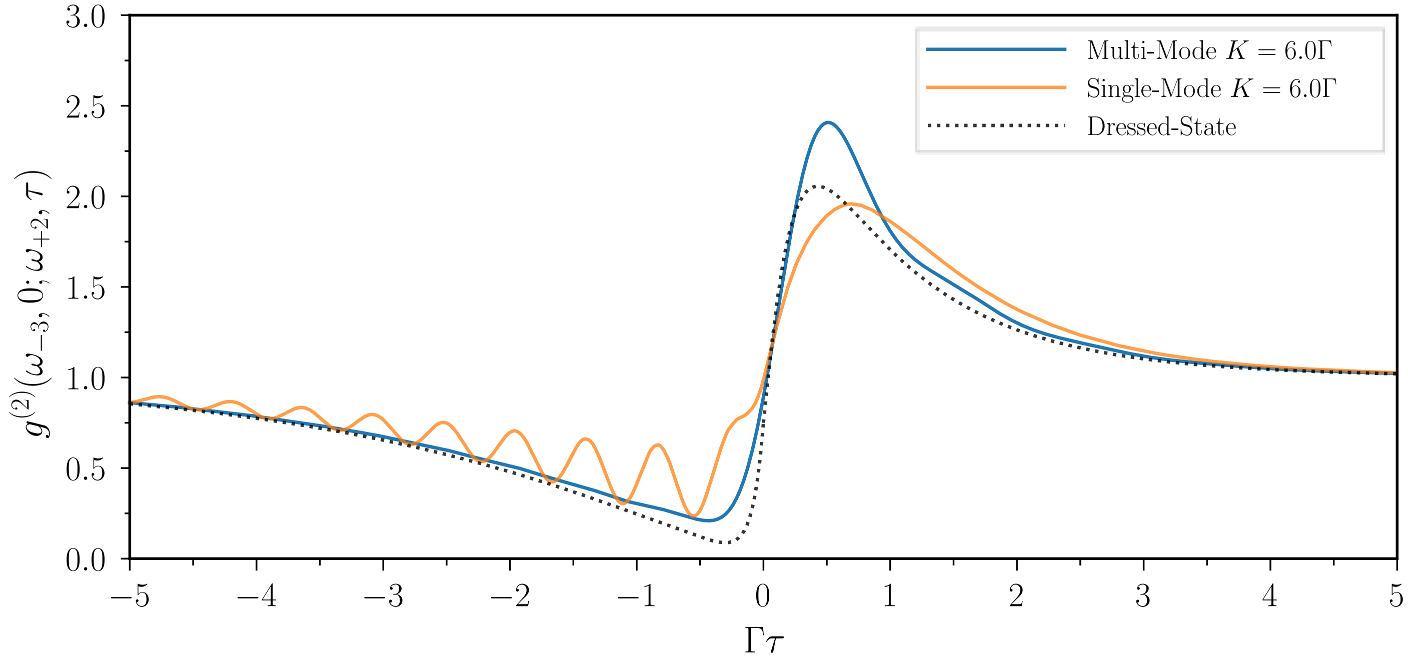


FREQUENCY-FILTERED CROSS-CORRELATIONS

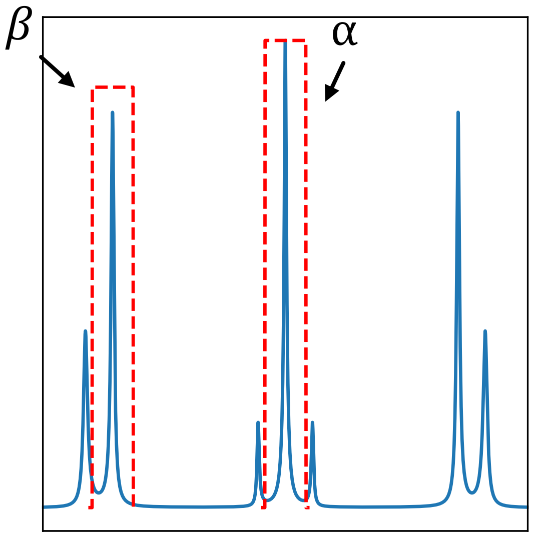


Interference of time-ordering
PRL **67**, 2443 (1991) PRA **45**, 8045 (1992)

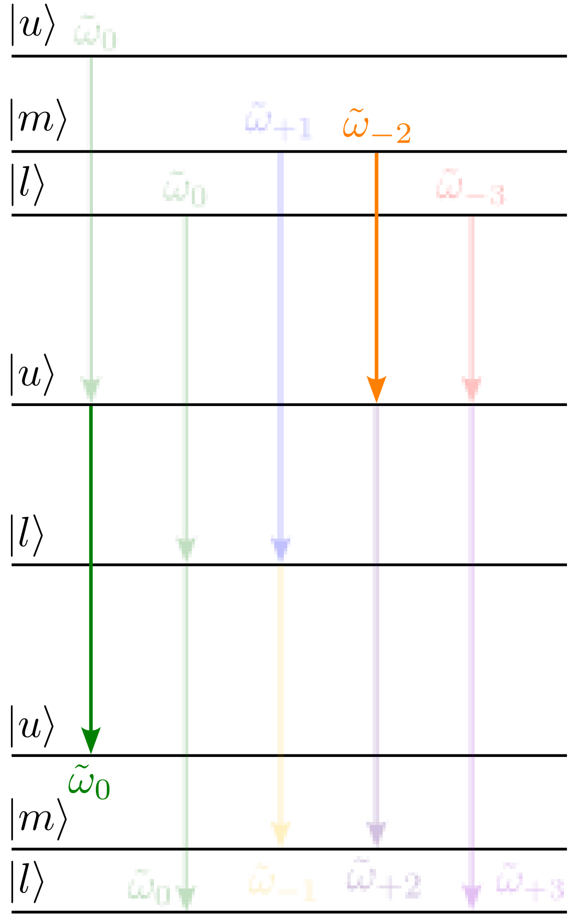


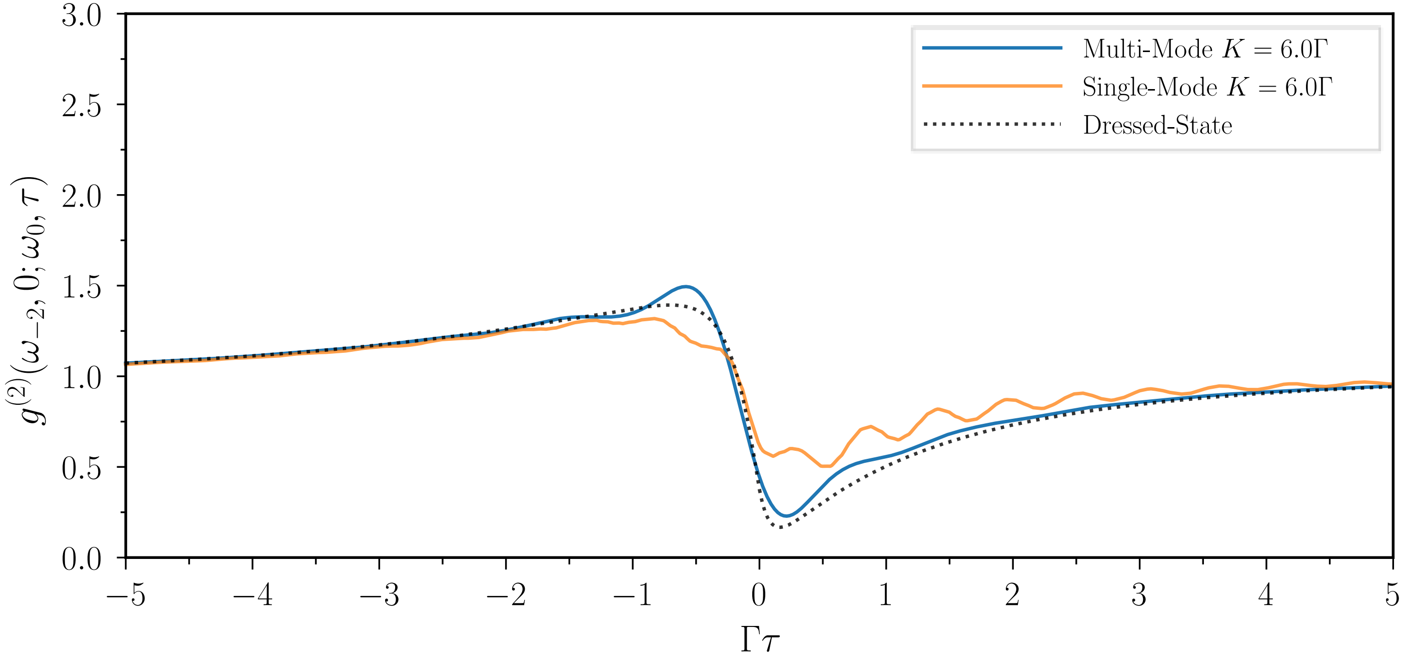


FREQUENCY-FILTERED CROSS-CORRELATIONS

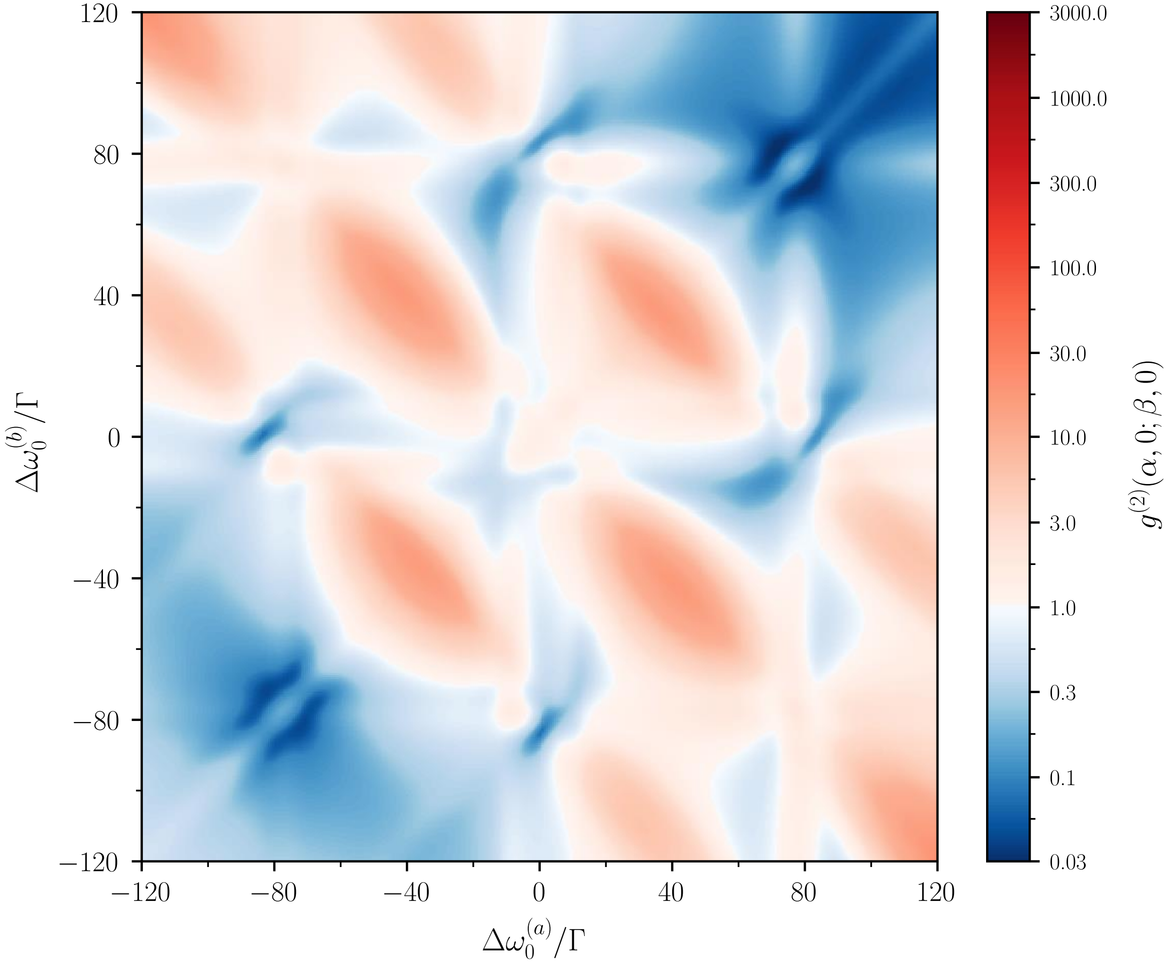


Interference of time-ordering
PRL **67**, 2443 (1991) PRA **45**, 8045 (1992)





“LANDSCAPE” OF PHOTON CORRELATIONS

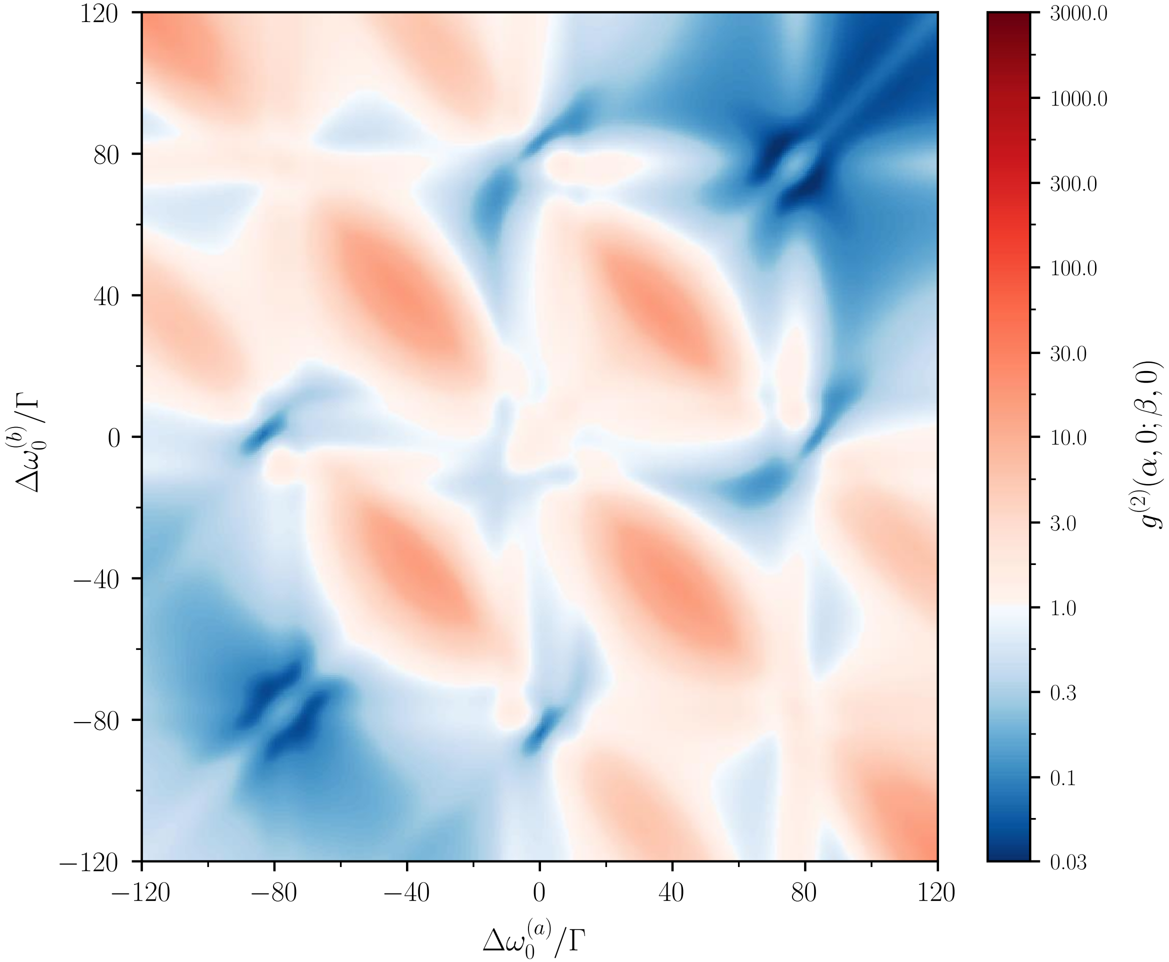


Single-mode filter with $K = 4\Gamma$

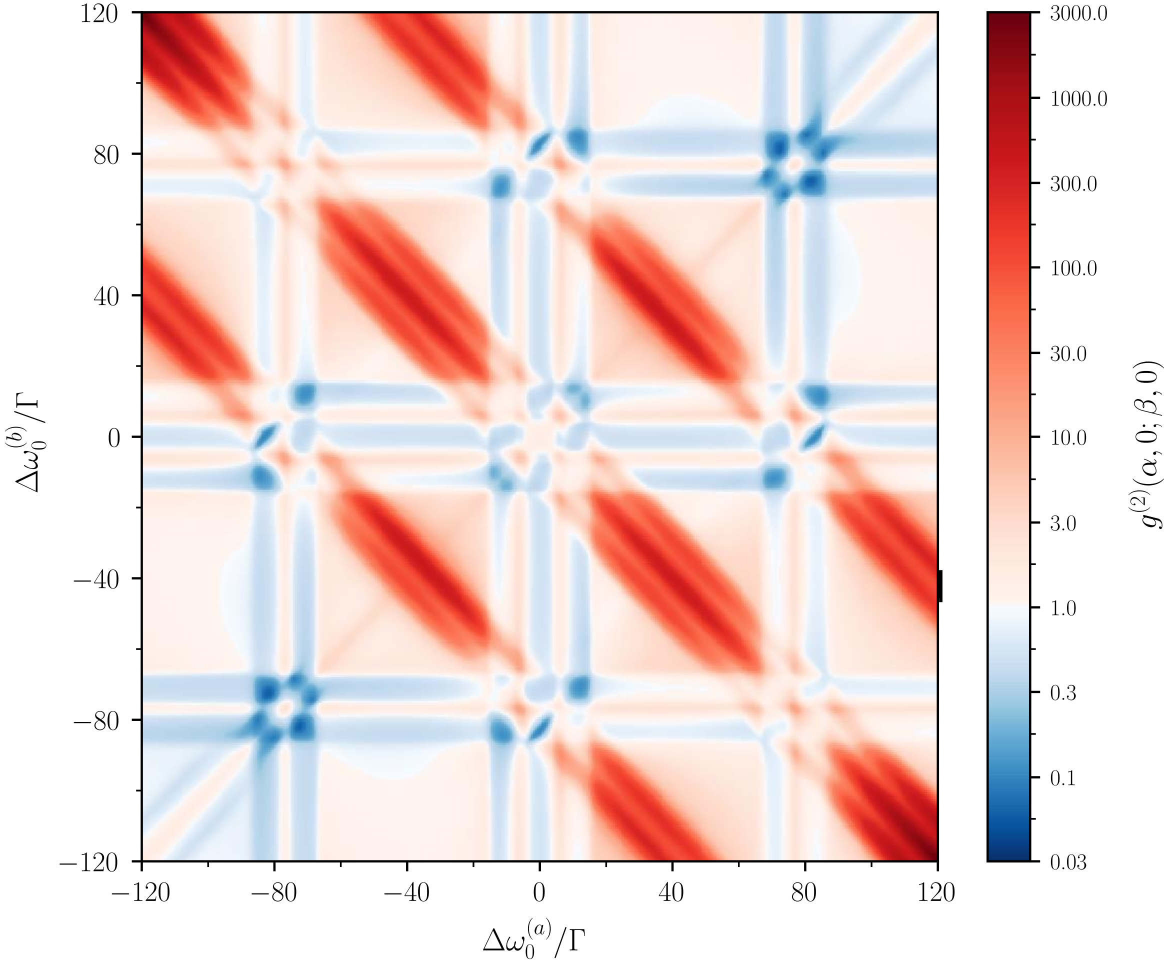
Initial correlation values

- Red – correlated / bunching
- White – uncorrelated / random
- Blue – anti-correlated / antibunched

“LANDSCAPE” OF PHOTON CORRELATIONS



Single-mode filter with $K = 4\Gamma$



Multi-mode array filter with $K = 4\Gamma$

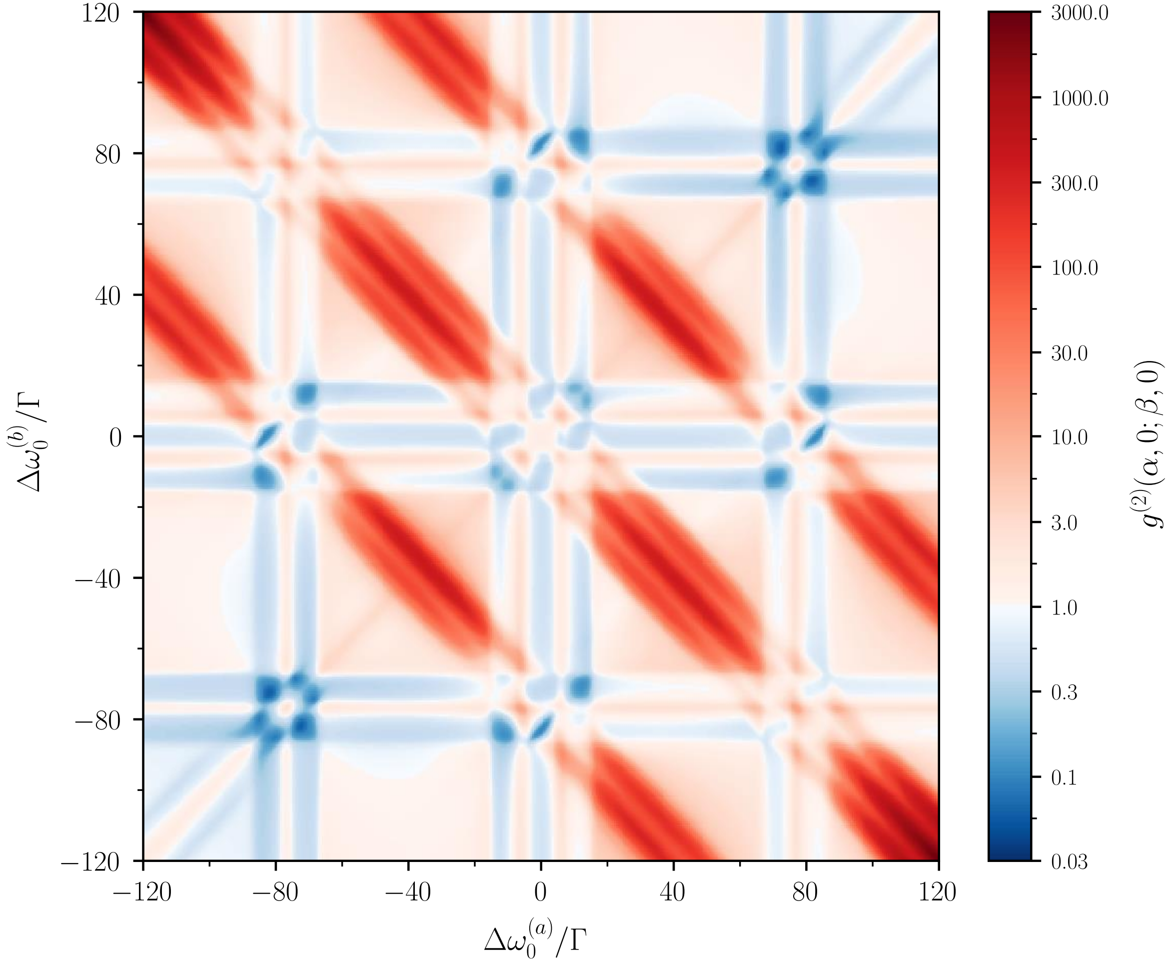
CONCLUSIONS

Three-level ladder type atom

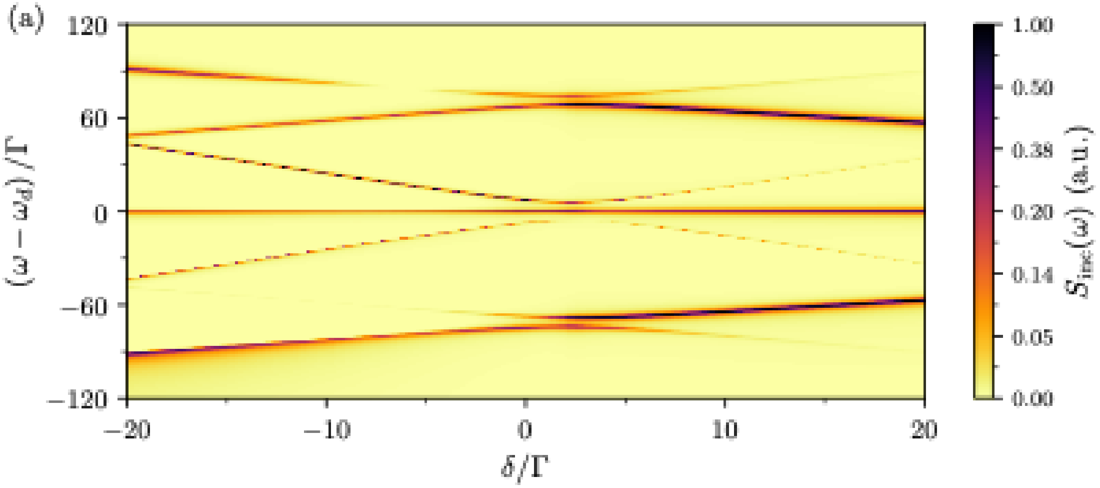
- Unique fluorescence spectrum
- New and interesting regions of photon correlations
- Phys. Rev. A **112**, 043701 (2025)

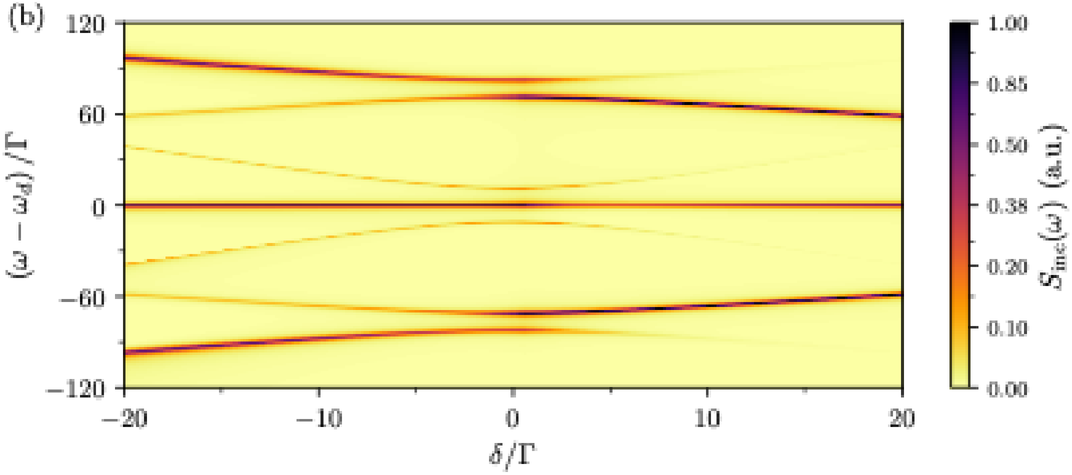
Multi-mode array filter

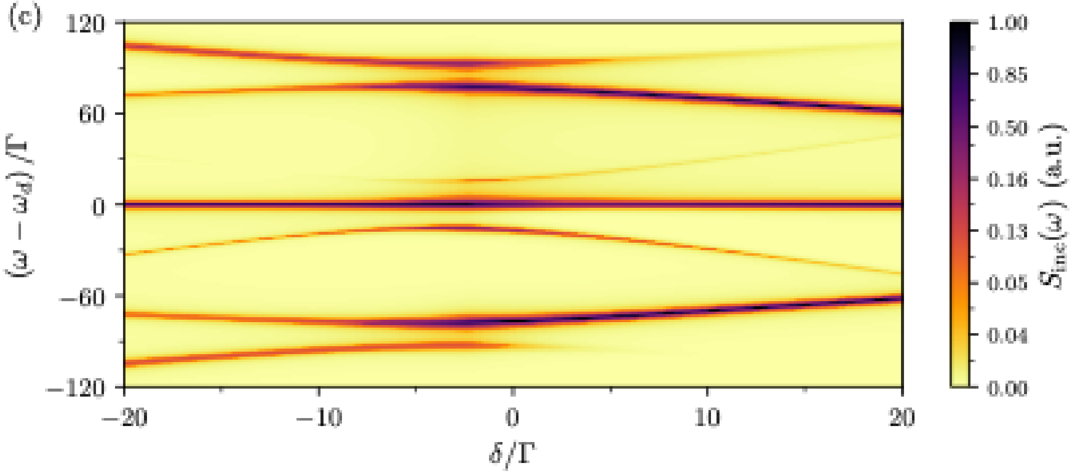
- Sharper frequency response can find for new regions of photon correlations
- A large system with an extremely efficient method of calculations
- *Maybe* an experimental nightmare
- Phys. Rev. A **100**, 023719 (2024)



FLUORESCENCE SPECTRUM – OFF RESONANCE

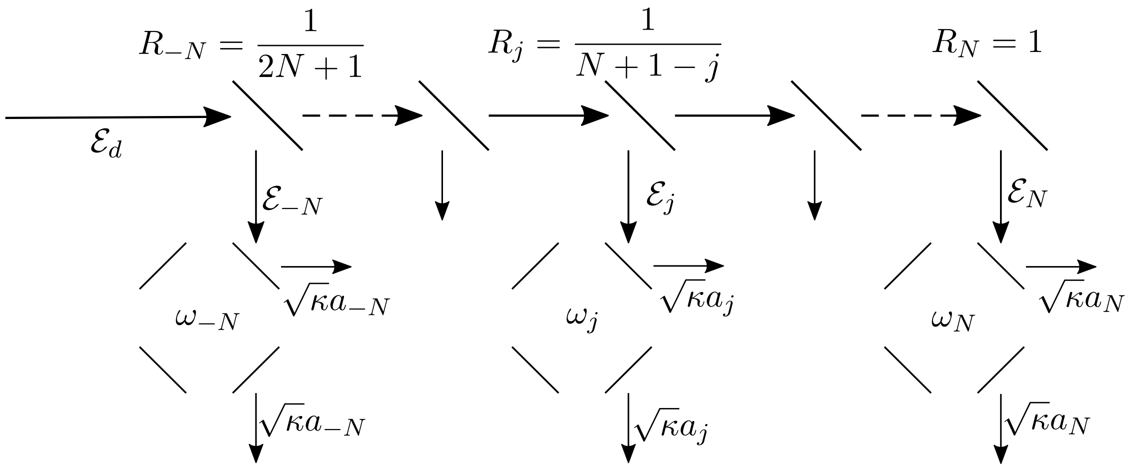




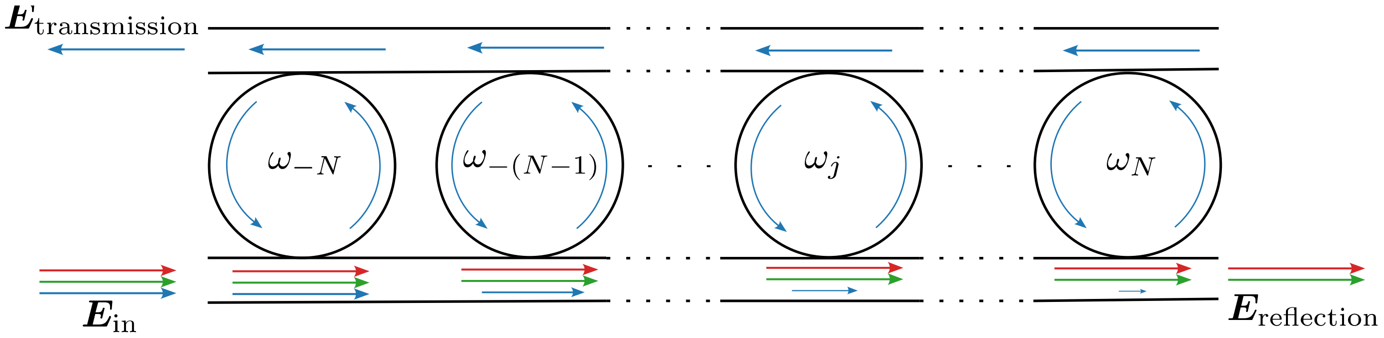


REALISATIONS OF THE MULTI-MODE ARRAY FILTER

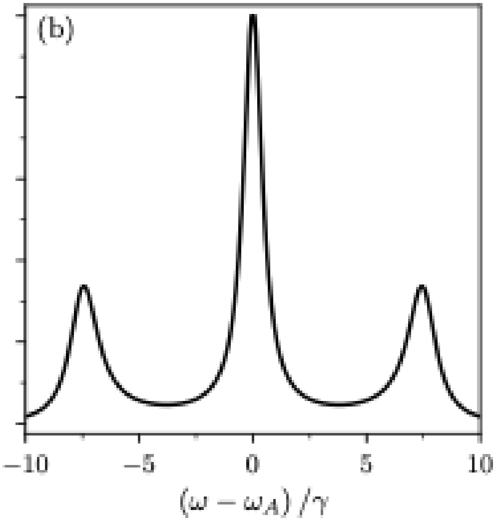
- Array of beam splitters, cascading into single-mode cavities

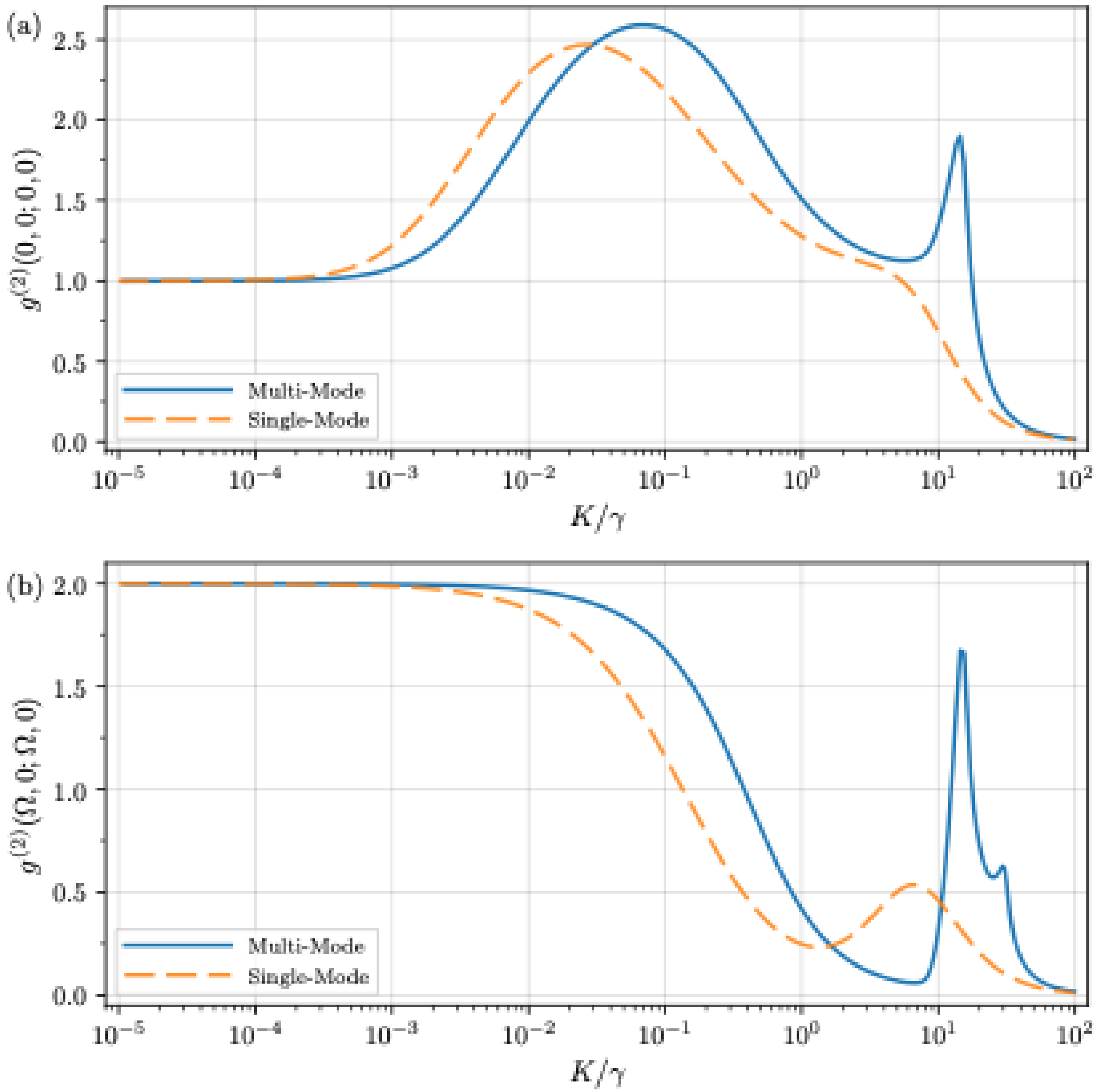


- Array of micro-ring-resonators



- Spatial light modulator into a single multi-mode cavity


(b)
−10
−5
0
5
10
$(\omega - \omega_A)/\gamma$


(a)
2.5
2.0
1.5
1.0
0.5
0.0
$g^{(2)}(0,0;0,0)$
Multi-Mode
Single-Mode
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
K/γ
(b)
2.0
1.5
1.0
0.5
0.0
$g^{(2)}(\Omega,0;\Omega,0)$
Multi-Mode
Single-Mode
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
K/γ

BACK-ACTION TERMS IN MOMENT EQUATIONS

CASCADED COUPLING

- Optical Bloch equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_-\rangle &= -\frac{\gamma}{2}\langle\sigma_-\rangle + i\frac{\Omega}{2}\langle\sigma_z\rangle\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_+\rangle &= -\frac{\gamma}{2}\langle\sigma_+\rangle - i\frac{\Omega}{2}\langle\sigma_z\rangle\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_z\rangle &= i\Omega\langle\sigma_-\rangle - i\Omega\langle\sigma_+\rangle - \gamma\left(\langle\sigma_z\rangle + 1\right)\end{aligned}$$

- First-order filter moments

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\sigma_-\rangle &= -\left(\frac{\gamma}{2} + \kappa + i\Delta\omega_j\right)\langle\sigma_-\rangle + i\frac{\Omega}{2}\langle\sigma_z\rangle\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\sigma_+\rangle &= -\left(\frac{\gamma}{2} + \kappa + i\Delta\omega_j\right)\langle\sigma_+\rangle - i\frac{\Omega}{2}\langle\sigma_z\rangle \boxed{-\frac{1}{2}\mathcal{E}_j\left(\langle\sigma_z\rangle + 1\right)}\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\sigma_z\rangle &= i\Omega\langle\sigma_-\rangle - i\Omega\langle\sigma_+\rangle - \left(\gamma + \kappa + i\Delta\omega_j\right)\langle\sigma_z\rangle \boxed{-\gamma\langle a_j\rangle + \mathcal{E}_j\langle\sigma_-\rangle}\end{aligned}$$

TWO-WAY COUPLING

$$H_I = i\hbar\sum_{j=-N}^{N}\left(\mathcal{E}_j^* a_j\sigma_+ - \mathcal{E}_j a_j^\dagger\sigma_-\right)$$

- Optical Bloch equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_-\rangle &= -\frac{\gamma}{2}\langle\sigma_-\rangle + i\frac{\Omega}{2}\langle\sigma_z\rangle + \sum_{j=-N}^{N}\mathcal{E}_j^*\langle a_j\sigma_z\rangle\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_+\rangle &= -\frac{\gamma}{2}\langle\sigma_+\rangle - i\frac{\Omega}{2}\langle\sigma_z\rangle + \sum_{j=-N}^{N}\mathcal{E}_j\langle a_j^\dagger\sigma_z\rangle\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_z\rangle &= i\Omega\langle\sigma_-\rangle - i\Omega\langle\sigma_+\rangle - \gamma\left(\langle\sigma_z\rangle + 1\right) + 2\sum_{j=-N}^{N}\mathcal{E}_j^*\langle a_j\sigma_+\rangle + 2\sum_{j=-N}^{N}\mathcal{E}_j\langle a_j^\dagger\sigma_-\rangle\end{aligned}$$

- First-order filter moments

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\sigma_-\rangle &= -\left(\frac{\gamma}{2} + \kappa + i\Delta\omega_j\right)\langle\sigma_-\rangle + i\frac{\Omega}{2}\langle\sigma_z\rangle + \sum_{k=-N}^{N}\mathcal{E}_k^*\langle a_j a_k\sigma_z\rangle\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\sigma_+\rangle &= -\left(\frac{\gamma}{2} + \kappa + i\Delta\omega_j\right)\langle\sigma_+\rangle - i\frac{\Omega}{2}\langle\sigma_z\rangle \boxed{-\frac{1}{2}\mathcal{E}_j\left(\langle\sigma_z\rangle + 1\right)} + \sum_{k=-N}^{N}\mathcal{E}_k^*\langle a_k^\dagger a_j\rangle\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle a_j\sigma_z\rangle &= i\Omega\langle\sigma_-\rangle - i\Omega\langle\sigma_+\rangle - \left(\gamma + \kappa + i\Delta\omega_j\right)\langle\sigma_z\rangle \boxed{-\gamma\langle a_j\rangle + \mathcal{E}_j\langle\sigma_-\rangle}\\ &\quad + \text{something else awful...}\end{aligned}$$